19. On Quasi-Fibrations over Spheres

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1. Let X be a CW-complex of the form $S^k \cup_\alpha e^n \cup_\beta e^{n+k}$. X is called a quasi-fibration over S^n if there exists a map

$$p : (X, S^k) \to (S^n, pt)$$

which induces homotopy isomorphisms. On the other hand we have the notion of k-spherical fibrations over S^n in the sense of Hurewicz.

Let $q : E \to S^n$ be a k-spherical fibration so that it is known that the pair (E, S^k) has the homotopy type such as (X, S^k). It is clear that, if (X, S^k) has the homotopy type of a pair (E, S^k) X is a quasi-fibration over S^n. In this note we shall prove the following

Theorem 1.1. *For a CW-complex X of the form $S^k \cup_\alpha e^n \cup_\beta e^{n+k}$ $(n \geqq k+2 \geqq 4)$. Let $p : (X, S^k) \to (S^n, pt)$ be a quasi-fibration. Then the pair (X, S^k) has the homotopy type of a pair of a k-spherical fibration over S^n.*

Remark. Probably, the condition $n \geqq k+2$ can be removed. Let $\tilde{\alpha} \in \pi_n(S^k \cup_\alpha e^n, S^k)$ be the generator which $\partial(\tilde{\alpha}) = \alpha$, let $\iota_k \in \pi_k(S^k)$ be the generator and let $i : S^k \to S^k \cup_\alpha e^n$ and $j : S^k \cup_\alpha e^n \to (S^k \cup_\alpha e^n, S^k)$ be the inclusions respectively.

For the proof of theorem we need following lemmas.

Lemma 1.2. *The pair (X, S^k) $(n \geqq k+2 \geqq 4)$ has the homotopy type of a pair of a k-spherical fibration over S^n if and only if*

$$j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r,$$

where $[,]_r$ denotes the relative Whitehead product.

Lemma 1.3. *Let $p : (X, S^k) \to (S^n, pt)$ be a quasi-fibration $(n \geqq k + 2 \geqq 4)$. Then we have $j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r$.*

It is obvious that Theorem 1.1 follows from lemmas.

Moreover, Theorem 2.1 in [1] shows that the existence of a quasi-fibration follows from the condition $j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r$. Hence we have

Collorary 1.4. *For $X = S^k \cup_\alpha e^n \cup_\beta e^{n+k}$ $(n \geqq k+2 \geqq 4)$, X has the homotopy type of the total space of a k-spherical fibration over S^n if and only if $j_*(\beta) = \pm [\tilde{\alpha}, \iota_k]_r$, or there exists a quasi-fibration $p : (X, S^k) \to (S^n, pt)$.*

2. In this section we shall give the proofs of lemmas. First we prove Lemma 1.3. Let $Q : S^k \cup_\alpha e^n \to S^n$ be the natural collapsing map. By a theorem of Blaker-Massy we know that