On the Existence of Quasiperiodic Solutions of Nonlinear Hyperbolic Partial Differential Equations

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1. Introduction.

In this note we shall consider a global property, that is, the quasi-periodic property, of the solutions of the following quasilinear one dimensional wave equation with dissipative term \( \alpha u_t \), where \( \alpha \) is a constant:

\[
M(u) = u_{tt} - u_{xx} + \alpha u_t = h(x, t, u, u_x, u_t),
\]

where \( h \) is quasiperiodic with basic frequencies \( \omega_1, \ldots, \omega_m \) in \( t \). We shall show the existence of such quasiperiodic solutions of the form (1) that have the same basic frequencies as \( h \) and satisfy the boundary conditions \( u(0, t) = u(\pi, t) = 0 \). These solutions are classical solutions.

The case \( m = 1 \) is the periodic case and was already solved by Rabinowitz [1], [2]. Especially, in [2] equation is strictly nonlinear.

2. Notations and definitions.

Definition. \( f(x, t) \) is called quasiperiodic with basic frequencies \( \omega_1, \ldots, \omega_m \) in \( t \), if there exists a function \( F(x, \theta_1, \ldots, \theta_m) \) such that \( f(x, t) = F(x, \omega t, \ldots, \omega_m t) \), where \( F(x, \theta_1, \ldots, \theta_m) \) is a continuous function of period \( 2\pi \) in \( \theta_1, \ldots, \theta_m \). Basic frequencies \( \omega_1, \ldots, \omega_m \) are real numbers.

We shall denote by \( \mathcal{F}^k(\omega_1, \ldots, \omega_m) \) the class of \( f(x, t) \) for which \( \mathcal{F}(x, \theta_1, \ldots, \theta_m) \) is \( C^k \)-class in \( x, \theta_1, \ldots, \theta_m \) and by \( \mathcal{F}^k(\omega_1, \ldots, \omega_m) \subset \mathcal{F}^k(\omega_1, \ldots, \omega_m) \) the class of \( f(x, t) \) which is \( 2\pi \)-periodic in \( x(1 \leq k \leq \infty) \).

Every \( f(x, t) \in \mathcal{F}^k \) is expanded in the Fourier series if \( k \geq 1 \):

\[
f(x, t) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^m} f_{jk} e^{ijx} e^{i\omega jt}.
\]

We introduce the norms in \( F^k \) by \( \| f \| = \sum \| f_{jk} \| \) and

\[
\| f \| = \| f \| + \| f_x \| + \| f_t \|.
\]

Now we assume that \( h(x, t, p, q, r) \) is in the form

\[
f(x, t) + g(x, t, p, q, r), f(x, t) \equiv 0.
\]

Then we can represent \( g(x, t, p, q, r) \) in the form \( G(x, \omega_1 t, \ldots, \omega_m t, p, q, r) \), where \( G(x, \theta_1, \ldots, \theta_m, p, q, r) \) is continuous and \( 2\pi \)-periodic in \( \theta_1, \ldots, \theta_m \).

Further we assume that \( f(x, t) \) and \( g(x, t, u, u_x, u_t) \) vanish at the boundary \( x = 0, x = \pi \).

3. The existence of quasiperiodic solutions.

3.1. At first we consider the case where the forcing term \( h(x, t, u, u_x, u_t) \) does not depend on \( u, u_x, u_t \):

\[
M(u) = u_{tt} - u_{xx} + \alpha u_t = f(x, t).
\]