71. On the Integral of Cauchy-Stieltjes Type and I. I. Privalov's Fundamental Lemma. II

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3. **Fundamental Lemma 2.** In Fundamental Lemma 1, we have assumed that there exist one-sided derivatives: $F'_{\pm}(s_0)$. If we assume that L has a finite curvature at x_0, then without the existence of $F'_{\pm}(s_0)$ we can prove

Fundamental Lemma 2. *Suppose that $x(s)$ is twice continuously differentiable in the neighbourhood of s_0. For a fixed α, put*

$$z=x_0+\varepsilon e^{i\varphi_0}\cdot e^{i\alpha}(0<\alpha<\pi), \qquad z^*=x_0+\varepsilon^* e^{i\varphi_0}\cdot e^{-i\alpha^*}(0<\alpha^*<\pi),$$

where

(1) $x_0=x(s_0)$, $\varphi_0=\varphi(s_0)$,

(2) $\varepsilon^*\to 0$, $\alpha^*\to\alpha$ *as* $\varepsilon\to+0$ *in such a manner that* $\varepsilon^* e^{-i\alpha^*}=\varepsilon e^{-i\alpha}(1+0(\varepsilon))$.

Then, putting $\varepsilon e^{i\alpha}=x+iy$, following propositions hold;

(1) $$\lim_{\varepsilon\to+0}\{f(z)-f(z^*)\}=A \rightleftarrows \lim_{\varepsilon\to+0}\frac{1}{\pi}\cdot\int_{-h}^{+h}\frac{y}{(\sigma-x)^2+y^2}dF(s_0+\sigma)=A,$$

where A: a finite complex number, h: any fixed positive constant.

(2) *If $F(s)$ is continuous at $s=s_0$, then*

$$\lim_{\varepsilon\to+0}\left\{f(z)+f(z^*)-\frac{1}{\pi i}\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x(s)-x_0}\right\}=0 \rightleftarrows$$

$$\lim_{\varepsilon\to+0}\left\{\int_{-h}^{h}\frac{\sigma-x}{(\sigma-x)^2+y^2}dF(s_0+\sigma)-\int_{\varepsilon}^{h}\frac{1}{\sigma}d(F(s_0+\sigma)+F(s_0-\sigma)\right\}=0,$$

where h: any fixed positive constant.

(3) *If* $\alpha=\frac{\pi}{2}$, i.e. $x=0$, $y=\varepsilon$, *then next estimation holds:*

$$f(z)+f(z^*)-\frac{1}{\pi i}\cdot\int_{L_\varepsilon}\frac{e^{i\varphi}dF(s)}{x(s)-x_0}$$
$$=0\left(\int_0^h\frac{\varepsilon}{\sigma^2+\varepsilon^2}\,|d(F(s_0+\sigma)+F(s_0-\sigma))|\right)+0\left(\int_{-h}^{h}|dF(s_0+\sigma)|\right)+o(1)$$

as $\varepsilon\to+0$, where h: any fixed positive constant.

From this lemma, we can derive some important boundary behaviours of the integral of Cauchy-Stieltjes type. We begin with

Corollary 2. *Assume that the conditions in Fundamental Lemma 2 are satisfied. Then following propositions hold;*

(1) *If there exists the finite symmetric derivative at s_0:*