68. On Normal Approximate Spectrum. II

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1972)

1. Introduction. Suppose that a (bounded linear) operator T acts on a Hilbert space $\mathfrak{H}$. A complex number λ is an *approximate propervalue* of T if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(*) \qquad \|(T-\lambda)x_n\| \to 0 \qquad (u \to \infty).$$

The set of all approximate propervalues of T is called the *approximate spectrum* $\pi(T)$ of T. According to Kasahara and Takai [8], an approximate propervalue λ of T is called *normal* if λ satisfies furthermore

$$(**) \qquad \|(T-\lambda)^* x_n\| \to 0 \qquad (n \to \infty).$$

The set $\pi_n(T)$ of all normal approximate propervalues of T is called the *normal approximate spectrum* of T. Several equivalent conditions which give the normal approximate spectra are discussed in [4] and [8].

In the present note, we shall concern with some additional properties of the normal approximate spectra of operators. In § 2, we shall give a theorem of Hildebrandt [7; Satz 2 (ii)] and observe its consequences. A theorem of Arveson [1; Theorem 3.1.2] follows at once. A theorem of Stampfli [9] is improved. In § 3, we shall show that a spectraloid is finite in the sense of Williams [10]. In § 4, we shall discuss a variant of a proposition of Bunce [3; Proposition 6].

2. Consequences of Hildebrandt's theorem. Hildebrandt [7] stated without proof the following theorem:

Theorem 1 (Hildebrandt). *If λ belongs to $\partial W(T)$ and $\pi(T)$, then $\lambda \in \pi_n(T)$, where $\partial W(T)$ is the frontier of the numerical range*

$$(1) \qquad W(T) = \{(Tx \mid x) ; \|x\| = 1\}.$$

Proof. We can assume that $\lambda = 0$ and $\operatorname{Re} T \geqq 0$ where

$$(2) \qquad \operatorname{Re} T = \frac{1}{2}(T + T^*).$$

Then we have

$$|(Tx_n \mid x_n)| \leqq \|Tx_n\| \to 0 \qquad (n \to \infty),$$

so that we have

$$(\operatorname{Re} Tx_n \mid x_n) \to 0 \qquad (n \to \infty).$$

Let A be the (positive) square-root of $\operatorname{Re} T$. Then we have

$$\|Ax_n\|^2 = (A^2 x_n \mid x_n) = (\operatorname{Re} Tx_n \mid x_n) \to 0 \qquad (n \to \infty).$$

Therefore we have

$$\|\operatorname{Re} Tx_n\| = \|A^2 x_n\| \to 0 \qquad (n \to \infty),$$