64. On $u_t = u_{xx} - F(u_x)$ and the Differentiability of the Nonlinear Semi-Group Associated with it

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1. Introduction. Suggested by a problem which has to do with the *burning of gas in a rocket* (see Forsythe and Wasow [4], p. 141), we consider in the present paper the following problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - F\left(\dfrac{\partial u}{\partial x}\right) & \text{in } (-\pi, \pi)\times(0,\infty), \\ u(-\pi, t) = u(\pi, t) & \text{in } (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } (-\pi, \pi), \end{cases}$$

where F is a *continuous* function on R^1 such that $F(0)=0$. We shall prove the existence and the uniqueness theorem[1] for solutions of (1.1) by studying the *differentiability of the nonlinear contraction semi-group* on $C_{2\pi}[-\pi,\pi] \subset L^\infty(-\pi,\pi)$, associated with (1.1), which is generated in the sense of Theorem I of Crandall and Liggett [3]; here $C_{2\pi}[-\pi,\pi]$ is the Banach space of all real-valued continuous functions f on $[-\pi,\pi]$ satisfying $f(-\pi)=f(\pi)$, endowed with the norm $\|\cdot\|_\infty$ induced by $L^\infty(-\pi,\pi)$.

2. Our result reads:

Theorem. *Assume that*

$$(2.1)\qquad u_0, \frac{du_0}{dx} \in C_{2\pi}[-\pi,\pi] \quad \textit{and} \quad \frac{d^2u_0}{dx^2} \in L^\infty(-\pi,\pi).$$

Then the equation (1.1) *has a unique solution* $u=u(x,t)$ *such that*

$$(2.2)\qquad u, \frac{\partial u}{\partial x} \in C([0,\infty); C_{2\pi}[-\pi,\pi]), \qquad \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \in L^\infty((-\pi,\pi)\times(0,\infty));$$

where $\partial u/\partial x, \partial^2 u/\partial x^2, \partial u/\partial t$ *denote the distribution derivatives of* $u \in \mathcal{D}'((-\pi,\pi)\times(0,\infty))$.

3. The uniqueness. We set, for $1 < p \leqslant \infty$,

$$\tau_p(f,g) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(\|f+\varepsilon g\|_p - \|f\|_p),\ f, g \in L^p(-\pi,\pi).\text{[2]}$$

Then, by Sato [14], § 6, we have

$$\tau_\infty(f,g) = \max_{x\in\{x;|f(x)|=\|f\|_\infty\}} (\operatorname{sgn} f(x))g(x), \quad f, g \in C_{2\pi}[-\pi,\pi], \quad f \neq 0,$$

and, for $1 < p < \infty$,

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1) Another approach to a similar problem is seen, for example, in Kruzhkov [12].

2) $\|\cdot\|_p = \|\cdot\|_{L^p(-\pi,\pi)}$.