92. *On the Index of Hypoelliptic Pseudo-differential Operators on R^n*

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§ 0. Introduction. The purpose of this paper is to prove that the index of a system P of pseudo-differential operators on R^n vanishes, if the symbol $\sigma(P)(x,\xi)$ is slowly varying in the sense of Grushin [1] and satisfies the condition which is a modification of Hörmander's condition for the existence of parametrix (cf. Hörmander [3] and Šubin [8]).

We shall denote by $S^m_{\rho,\delta}$, $0\leqq\delta<\rho\leqq 1$, $-\infty<m<\infty$, the set of all C^∞-symbols $p(x,\xi)$ defined in $R^n_x\times R^n_\xi$, which satisfy for multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$,

(0.1) $$|p^{(\beta)}_{(\alpha)}(x,\xi)|\leqq C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

for some constants $C_{\alpha,\beta}$, where $p^{(\beta)}_{(\alpha)}(x,\xi)=D^\alpha_x\partial^\beta_\xi p(x,\xi)$, $D^\alpha_x=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}$, $\partial^\beta_\xi=(\partial/\partial\xi_1)^{\beta_1}\cdots(\partial/\partial\xi_n)^{\beta_n}$. We set $S^\infty_{\rho,\delta}=\bigcup_m S^m_{\rho,\delta}$ and $S^{-\infty}_{\rho,\delta}=\bigcap_m S^m_{\rho,\delta}$. For a symbol $p(x,\xi)\in S^m_{\rho,\delta}$ we define a pseudo-differential operator $p(X,D_x)$ by

(0.2) $$p(X,D_x)u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}(\xi)$ denotes the Fourier transform of a rapidly decreasing function $u(x)$ defined by $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$.

We say that a symbol $p(x,\xi)$ ($\in S^m_{\rho,\delta}$) is slowly varying, if the estimate (0.1) holds for a bounded function $C_{\alpha,\beta}(x)$ such that $C_{\alpha,\beta}(x)\to 0$ as $|x|\to\infty$ in case $\alpha\neq 0$ (cf. Grushin [1], p. 206).

Our main theorem is stated as follows:

Main theorem. *Let $p(x,\xi)=(p_{jk}(x,\xi))$ be an $l\times l$ matrix of symbols $p_{jk}(x,\xi)$ of class $S^m_{\rho,\delta}$, $m>0$, which are slowly varying. Assume that there exist positive constants c_0, c_1 and $0<\tau\leqq 1$ such that $(p(x,\xi)-\zeta I)^{-1}$ exists on*

$$\Xi_\xi=\{\zeta\in \boldsymbol{C};\ \mathrm{dis}\,(\zeta,(-\infty,0])\leqq c_0(1+|\xi|)^{\tau m}\}$$

and the estimate of the form

(0.3) $$\|p^{(\beta)}_{(\alpha)}(x,\xi)(p(x,\xi)-\zeta I)^{-1}\|\leqq C_{\alpha,\beta}(x)(1+|\xi|)^{\delta|\alpha|-\rho|\beta|}$$

holds uniformly on Ξ_ξ, where $\|\cdot\|$ denotes a matrix norm, and $C_{\alpha,\beta}(x)$ is a bounded function which tends to zero as $|x|\to\infty$ in case $\alpha\neq 0$. Then the operator $P=p(X,D_x)$ as the map from $L^2=L^2(R^n)$ into itself with the domain $\mathcal{D}(P)=\{u\in L^2;\ Pu\in L^2\}$ is Fredholm type and we have