# 88. *Abelian Projections over a von Neumann Subalgebra*

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**1.** In the theory of von Neumann algebras the notions of abelian and minimal projections play important roles.

In this note we shall introduce that a projection of a von Neumann algebra is minimal or abelian over a von Neumann subalgebra. These notions are generalizations of minimal and abelian projections, and they are same if the von Neumann subalgebra is included in the center. In this note, we shall prove that some elementary properties of minimal and abelian projections are preserved under our generalizations. Furthermore, we shall obtain certain conditions that the support projection of a normal state is minimal or abelian over certain von Neumann subalgebras.

We shall use the terminology due to Dixmier [2] throughout the note without further explanations.

**2.** In the sequel, let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$. Denote $\mathcal{B}^c$ the relative commutant $\mathcal{B}' \cap \mathcal{A}$ of $\mathcal{B}$ and by $\bar{E}$ the $\mathcal{B}$-support of a projection $E$ in $\mathcal{A}$, that is, $\bar{E}$ is the infimum of projections in $\mathcal{B}$ which dominate $E$.

**Lemma 1.** *If $\bar{E}P$ is a projection for a projection $P$ in $\mathcal{B}$ and a projection $E$ in $\mathcal{B}^c$, then $\bar{E}P = \overline{EP}$.*

**Proof.** If $\bar{E}P \neq \overline{EP}$, then there exists a projection $Q$ in $\mathcal{B}$ such that

$$\bar{E}P > Q \geqq EP.$$

Then $Q + \bar{E}(1-P)$ is a projection in $\mathcal{B}$ and

$$\bar{E} > Q + \bar{E}(1-P) \geqq E,$$

which is a contradiction.

**Definition 1.** A projection $E \in \mathcal{A}$ is called to be *abelian over* $\mathcal{B}$ if $E \in \mathcal{B}^c$ and, for any projection $P \in \mathcal{A}$ which is dominated by $E$, there exists a projection $Q \in \mathcal{B}$ such that $P = QE$.

**Remark.** If $\mathcal{B}$ is the center of $\mathcal{A}$, a projection abelian over $\mathcal{B}$ is abelian in the usual sense, cf. [2].

If $\mathcal{A}$ is an abelian von Neumann algebra, the notion that a projection is abelian over $\mathcal{B}$ is introduced by Dye [3].

**Lemma 2.** *If a projection $E \in \mathcal{A}$ is abelian over $\mathcal{B}$, then a projection $P \leqq E$ is written in $P = \bar{P}E$.*