108. On Exponential Semigroups. II

By Takayuki TAMURA and Thomas E. NORDAHL
University of California, Davis, California, U. S. A.

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1. Introduction. Tamura and Shafer proved in [3] the following:

Theorem 1. *If S is an exponential archimedean semigroup with idempotent, then S is an ideal extension of I by N where I is the direct product of an abelian group G and a rectangular band B and N is an exponential nil-semigroup.*

However, the converse is not necessarily true. For example, let $S=\{a, b, c, d\}$ be the semigroup of order 4 defined by $(x, y \in S)$

$$xy=y \text{ for } y\neq d \text{ and all } x;\ xd=a \text{ for } x\neq c;\ cd=b.$$

S is the ideal extension of a right zero semigroup $\{a, b, c\}$ by a null semigroup of order 2. Associativity of S is easily verified, but S is not exponential:

$$(cd)^2=b^2=b, \qquad c^2d^2=ca=a.$$

The purpose of this paper is to prove Theorem 2 which characterizes exponential ideal extensions of I by N, and to give an alternate proof of the fact that I is completely simple. See the definition of the used terminology in [3] and [1]. The notation may be different from that in [1].

Theorem 2. *S is an exponential archimedean semigroup with idempotent if and only if S is an ideal extension of the direct product $I=\Lambda\times G\times M$ of a left zero semigroup Λ, an abelian group G, and a right zero semigroup M by an exponential nil-semigroup N, with product determined by three partial homomorphisms $\varphi: N\setminus\{0\}\to M$, $\mathfrak{G}: N\setminus\{0\}\to G$, $\psi: N\setminus\{0\}\to \Lambda$ in the following manner. Let (λ, a, μ), $(\nu, b, \eta) \in \Lambda\times G\times M$, $s, t \in N\setminus\{0\}$.*

$$(2.1)\qquad \begin{cases} (\lambda, a, \mu)\cdot s=(\lambda, a(s\mathfrak{G}), s\varphi) \\ s\cdot(\lambda, a, \mu)=(\psi s, (s\mathfrak{G})a, \mu) \\ (\lambda, a, \mu)\cdot(\nu, b, \eta)=(\lambda, ab, \eta) \\ s\cdot t=\begin{cases} st & \text{if } st\neq 0 \text{ in } N \\ (\psi s, (s\mathfrak{G})(t\mathfrak{G}), t\varphi) & \text{if } st=0 \text{ in } N \end{cases} \end{cases}$$

2. Alternate proof of complete simpleness of I. In [3] Anderson's theorem on bicyclic subsemigroup was used, but we will derive primitiveness of idempotent elements. Assume that S is an exponential archimedean semigroup. Let e be an idempotent element of S and let $I=SeS$. Since $I\subseteq SaS$ for all $a \in S$, I is the kernel of S and hence I is simple. Let e and f be idempotents such that $ef=fe=f$. Now IeI