# 104. *A Pointwise Ergodic Theorem for Positive Bounded Operator*

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**1. Introduction and the theorem.** The purpose of this note is to prove a pointwise ergodic theorem for a positive bounded linear operator which generalizes those induced by non-singular measurable transformations and Markov processes with an invariant measure. Throughout this note, let $(X, \mathfrak{B}, m)$ be a finite measure space. We denote the norm and the operator norm in $L_p(X)$ by $\| \ \|_p$ $(1 \leqslant p \leqslant \infty)$. Let $T$ be a positive bounded linear operator defined on $L_1(X)$. (The positivity means that $Tf \geqslant 0$, if $f \geqslant 0$.) Put $S_n = \sum_{k=0}^{n-1} T^k$, where $T^0 = I$ (identity). In the sequel we assume that the operator $T$ satisfies the following conditions.

(A) There exists a constant $K > 0$ such that

$$\|(1/n)S_n\|_1 \leqslant K \quad \text{and} \quad \|(1/n)S_n\|_\infty \leqslant K (n = 1, 2, \cdots),$$

(B) $\lim_{n\to\infty} \|(T^n/n)f\|_1 = 0$ for any $f \in L_1(X)$ and $\lim_{n\to\infty} \|(T^n/n)f\|_\infty = 0$ for any $f \in L_\infty(X)$,

(C) If $f \geqslant 0$, $f \in L_1(X)$ and $\liminf_{n\to\infty} \|(S_n/n)f\|_1 = 0$, then $f = 0$.

We shall prove the following

**Theorem.** *Let $T$ be a positive bounded linear operator on $L_1(X)$. If the operator $T$ satisfies three conditions* (A), (B) *and* (C), *then a pointwise ergodic theorem holds for $T$, that is, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (T^k f)(x)$$

*exists almost everywhere for any $f \in L_1(X)$ and it is in $L_1(X)$.*

**Remark.** The operator in the theorem includes those induced by measure preserving transformations (the Birkhoff's pointwise ergodic theorem). Consider an operator induced by a non-singular measurable transformation. Then we have a pointwise ergodic theorem for the operator only if the operator satisfies the above condition (C). For the operator induced by a Markov process, there exists a finite invariant measure $\mu$ with $\mu \sim m$ if and only if the operator satisfies the above condition (C) [3]. The operator in the theorem includes a positive invertible operator $T$ with $\sup_{-\infty<n<\infty} \|T^n\|_1 < \infty$ and $\sup_{-\infty<n<\infty} \|T^n\|_\infty < \infty$.

**2. The proof of the theorem.**