149. On Quasi-primality of Submodules and of Ideals in Rings

By Hisao IZUMI
Ube Technical College

(Comm. by Kenjiro SHODA, M. J. A., Nov. 13, 1972)

W. E. Barnes [1] has given for rings a theory of the representation of an ideal as an intersection of primal ideals, and showed that, in any short reduced representation of an ideal by primal ideals with prime adjoints, the adjoints and the number of primal components are uniquely determined. As is well-known, in that case there exist no containment relations among the prime adjoints. In order to generalize the above results, we shall consider a representation of a submodule by quasi-primal submodules, and as a special case we obtain that any two short reduced representations of an ideal by primal ideals have the same number of primal components and the same McCoy's radicals of their adjoints in pairs, if there exist no containment relations among the McCoy's radicals of the adjoints of primal components.

Throughout this note, R is a noncommutative ring whose unity does not necessarily exist, and M is a right R-module. The term *ideals* mean *two-sided ideals*, and (x) means the principal ideal by an element x of R. For a subset S of R, we set $\bar{S}=\{x\in R\mid (x)^n\subseteq S$ for some positive integer $n\}$, and set $\tilde{S}=\bigcap_i\{P_i\mid P_i$ is a prime ideal and $P_i\supseteq S\}$. Hence $\tilde{S}$ is an ideal. For convenience, even if a subset S of R is not an ideal, $\tilde{S}$ is called the *McCoy's radical* of S. For all ordinal numbers α we define $\bar{S}^{(\alpha)}$ by induction as follows: $\bar{S}^{(1)}=\bar{S}$, if α is not a limit ordinal then $\bar{S}^{(\alpha)}=\overline{\bar{S}^{(\alpha-1)}}$, and if α is a limit ordinal then $\bar{S}^{(\alpha)}=\bigcup_{\beta<\alpha}\bar{S}^{(\beta)}$.

Definition 1. Let S be a subset of R. If $\bar{S}^{(\alpha)}$ is an ideal for some ordinal number α, S is called a *quasi-ideal.*

Definition 2. A submodule N of M is called a *primal submodule* if its *adjoint subset* $N^a=\{x\in R\mid N: x\supsetneqq N\}$ is an ideal, where $N: x$ means the submodule $\{m\in M\mid mRx\subseteq N\}$. A submodule N of M is called a *quasi-primal submodule* if the adjoint subset N^a is a quasi-ideal. Evidently a primal submodule is quasi-primal.

Lemma 1. *If an ideal A of R is contained in the set-union of finitely many semi-prime ideals Q_i, then A is contained in one of the Q_i.*

Proof. Suppose that $A\not\subseteq Q_i$ for every i, then there exist prime ideals P_i such that $P_i\supseteq Q$ and $P_i\not\supseteq A$. Hence $A\subseteq\bigcup_{i=1}^n Q_i\subseteq\bigcup_{i=1}^n P_i$. This contradicts the well-known McCoy's result.