# 148. *Iterated Loop Spaces*

By Yasutoshi NOMURA
College of General Education, Osaka University

(Comm. by Kenjiro SHODA, M. J. A., Nov. 13, 1972)

The aim of this note is to give conditions under which a space or a map can be de-looped $k$-times up to homotopy. The duals to Theorems 1 and 2 have been obtained by Berstein-Ganea [2]. Our basic lemma (Lemma 1) allows us to overcome the difficulty which arises in dualizing Theorem 3.3 of T. Ganea [4], thereby obtaining a de-looping theorem for a homotopy $\Omega^k S^k$-space (see Theorem 4).

**1. A basic lemma.** First we set up some notation and conventions. The spaces we consider are supposed to have the based homotopy type of $CW$-complexes. We denote the loop and suspension functors by $\Omega$ and $S$. Given a map $u: A \to B$, the fibre $\{(a, \gamma) \in A \times B^I; \gamma(0) = *, \gamma(1) = u(a)\}$ and the cofibre $B \cup_u CA$ are denoted by $E_u$ and $C_u$ respectively. The identity maps $\Omega^k X \to \Omega^k X$ and $S^k X \to S^k X$ yield the canonical adjointness maps $\varepsilon_k: S^k \Omega^k X \to X$ and $\eta_k: X \to \Omega^k S^k X$.

Now given a map $f: \Omega X \to Y$, introduce the homotopy commutative diagram

$$
\begin{array}{ccccccccc}
\Omega X & \xrightarrow{f} & Y & & & & & & \\
\downarrow{\scriptstyle \alpha'} & & \| & & & & & & \\
E_i & \longrightarrow & Y & \xrightarrow{i} & C_f & & & & \\
\downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \alpha} & & \| & & & & \\
\Omega X & \longrightarrow & E_{\varepsilon_1 q} & \longrightarrow & C_f & \xrightarrow{\varepsilon_1 q} & X & & \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow & & \| & & \\
\Omega X & \xrightarrow{-\Omega j} & \Omega C_{\varepsilon_1 q} & \longrightarrow & E_j & \longrightarrow & X & \xrightarrow{j} & C_{\varepsilon_1 q}
\end{array}
$$

in which the vertical maps are constructed as in p. 132 of [6] using the canonical homotopies, $i$ and $j$ are inclusions and $q: C_f \to S\Omega X$ the map pinching $Y$ to a point. Using the Blakers-Massey theorem (see e.g. Theorem 4.3 of [8]) we have

i) $(\beta\alpha) f \simeq \Omega j$,

ii) the construction of $\beta\alpha$ is functorial,

iii) if $f$ is $m$-connected, $m \geqq 1$, $X$ is 2-connected and $Y$ is $(n-1)$-connected, $n \geqq 1$, then $\beta\alpha$ is $[m + \min(m, n)]$-connected, $j$ $(m+1)$-connected and $C_{\varepsilon_1 q}$ is $\min(n, 2m+1)$-connected.

Iterating the process for $j$, we get

**Lemma 1.** *If $f: \Omega^k X \to Y$ is $m$-connected such that $X$ is $(k+1)$-*