# 8. *On Measurable Functions. II*

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In this part of the paper, some relations between the sets $\mathcal{H}$ and $\mathcal{G}$ stated in the introduction in Part I will be discussed.

**3. The set of all measurable functions. Assumption 3.1.** *$M$ is a non-empty set and $\mathcal{S}$ is a ring of subsets of $M$.*

For a topological additive group $K$, throughout this section we shall use the following notations:

1) Let $G=J=\{0\}$ be the topological additive group consisting of only one element and define the product of $0 \in G$ and $k \in K$ by $0 \cdot k=0 \in J$. Then the system $(M, G, K, J)$ becomes an integral system and this integral system is denoted by $\Lambda(K)$.[1]

2) $\mathcal{F}(K)$ is the total functional group of $\Lambda(K)$.

Then $(\mathcal{S}, \mathcal{F}(K), J)$ is an abstract integral structure.

3) $\mathcal{G}(K)$ is the integral closure of $K$ in $\mathcal{F}(K)$.

4) $\mathcal{G}_0(K)$ is the subgroup of $\mathcal{F}(K)$ generated by $\mathcal{S}K$.

Then $\mathcal{G}(K)$ is the $\mathcal{F}(K)$-completion of the closure of $\mathcal{G}_0(K)$ in $\mathcal{F}(K)$.

5) $\mathcal{V}(K)$ is the system of neighbourhoods of $0 \in K$ and $\tilde{V}=\{f \mid f \in \mathcal{F}(K), f(M) \subset V\}$ for each $V \in \mathcal{V}(K)$.

Then $\{\tilde{V} \mid V \in \mathcal{V}(K)\}$ is a base of the system of neighbourhoods of $0 \in \mathcal{F}(K)$.

Now we can state a property of $\mathcal{G}(K)$ corresponding to Theorem 2.1 in [1].

**Theorem 3.1.** *Let $K_i$, $i=1,2,\cdots,n$, be topological additive groups. Let $D$ be a subspace of the product space $\prod_{i=1}^{n} K_i$ and $\varphi$ a uniformly continuous map of $D$ into a topological additive group $K$. Then, for $f_i \in \mathcal{G}(K_i)$, $i=1,2,\cdots,n$, such that $(f_1(x),\cdots,f_n(x)) \in D$ for each $x \in M$, and for the map $f$ of $M$ into $K$ defined by $f(x)=\varphi(f_1(x), \cdots, f_n(x))$ for each $x \in M$, it holds that $f \in \mathcal{G}(K)$.*

**Proof.** Let $X$ be an element of $\mathcal{S}$. It suffices to show that $Xf \in \overline{\mathcal{G}_0(K)}$ or equivalently that $(Xf+\tilde{V}) \cap \mathcal{G}_0(K) \neq \phi$ for any $V \in \mathcal{V}(K)$. The uniform continuity of $\varphi$ implies the existence of $V_i \in \mathcal{V}(K_i)$, $i=1,2,\cdots,n$, satisfying the condition: $\varphi(x_1,\cdots,x_n)-\varphi(y_1,\cdots,y_n) \in V$

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1) The topological additive groups $G$ and $J$ play no essential role here. These groups are introduced only for the sake of the definitions of $\mathcal{F}(K)$, $\mathcal{G}(K)$, etc. Therefore, $G$ and $J$ may be replaced by any other groups such that $(M, G, K, J)$ becomes an integral system.