7. *On Measurable Functions. I*

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1. Introduction. An integral structure Γ was defined and an integral σ with respect to Γ was discussed in the author [3]. Let $\Lambda=(M, G, K, J)$ be an integral system and $\mathcal{S}$ a measurable ring of Λ. Then the fundamental integral structure $\Gamma=(\Lambda; \mathcal{S}, \mathcal{G}, Q)$ is determined by Λ and $\mathcal{S}$. Theorem 1 in [3] states that there exists a unique integral with respect to Γ provided that J is Hausdorff and complete. The set $\mathcal{G}$ of all integrands is the integral closure of K in the total functional group $\mathcal{F}$ of Λ with respect to the abstract integral structure $(\mathcal{S}, \mathcal{F}, J)$.

In this part of the paper, we shall define the measurability of a function $f \in \mathcal{F}$ and state some properties of measurable functions. Some relations between the set $\mathcal{H}$ of all measurable functions and the set $\mathcal{G}$ of all integrands will be discussed in Part II.

2. Measurable functions. Assumption 2.1. *M is a set and $\mathcal{S}$ is a ring of subsets of M.*

A map f of M into a topological space K is *measurable* if $f^{-1}(O) \cap X \in \mathcal{S}$ for any open set O in K and for any $X \in \mathcal{S}$.

Proposition 2.1. *Let N be a set and $\mathcal{A}$ a set of subsets of N. Let f be a map of M into N such that $f^{-1}(Y) \cap X \in \mathcal{S}$ for any $Y \in \mathcal{A}$ and $X \in \mathcal{S}$. Then we have*

1) *$f^{-1}(Y) \cap X \in \mathcal{S}$ for any element Y of the ring generated by $\mathcal{A}$ and for any $X \in \mathcal{S}$.*

2) *Assume that $\mathcal{S}$ is a pseudo-σ-ring. Then $f^{-1}(Y) \cap X \in \mathcal{S}$ for any element Y of the σ-ring generated by $\mathcal{A}$ and for any $X \in \mathcal{S}$.*

Proof. Putting $\mathcal{T}=\{Y \mid Y \subset N, f^{-1}(Y) \cap X \in \mathcal{S}$ for any $X \in \mathcal{S}\}$, we have $\mathcal{A} \subset \mathcal{T}$. For $Y, Z \in \mathcal{T}$ and for any $X \in \mathcal{S}$, it holds that $f^{-1}(Y-Z) \cap X=(f^{-1}(Y)-f^{-1}(Z)) \cap X=(f^{-1}(Y) \cap X)-(f^{-1}(Z) \cap X) \in \mathcal{S}$ and hence $Y-Z \in \mathcal{T}$. Analogously, $Y \cup Z \in \mathcal{T}$ for any $Y, Z \in \mathcal{T}$. Since $\phi \in \mathcal{T}$, it follows that $\mathcal{T}$ is a ring containing $\mathcal{A}$. Hence $\mathcal{T}$ contains the ring generated by $\mathcal{A}$ and thus 1) is proved. If $\mathcal{S}$ is a pseudo-σ-ring, we have $\bigcup_{i=1}^{\infty} Y_i \in \mathcal{T}$, for $Y_i \in \mathcal{T}$, $i=1,2,\cdots$, and this implies that $\mathcal{T}$ is a σ-ring containing $\mathcal{A}$. Thus 2) is proved.

Corollary 1. *Let K be a topological space and suppose that a map f of M into K is measurable. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be the ring and the σ-ring, respectively, generated by the set of all open sets in K. Then we have*

1) *$f^{-1}(Y) \cap X \in \mathcal{S}$ for any $Y \in \mathcal{T}_0$ and $X \in \mathcal{S}$.*