3. *A Proof of Negative Answer to Hilbert's 10th Problem*

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0. Recently, the effective methods for Diophantine equations make a rapid progress.

A. Baker gave an effective procedure for the existence of integer solutions of some kinds of Diophantine equations in [1].

In his paper [2], Ju. B. Matijasevič proved the unsolvability of Hilbert's 10th problem by using the results of Julia Robinson, M. Davis and H. Putnum in [3], [4] and [5].

In the present note, we shall give a short proof of the negative solution of Hilbert's 10th problem. That is, we lead to the unsolvability of the problem directly from the following result of Davis [3]:

Every recursively enumerable set S can be expressed in the form,

(∗) $x \in S \equiv (\exists y)(\forall k)_{k<y}(\exists z_1)\cdots(\exists z_m)[P(x, y, k, z_1, \cdots, z_m)=0]$,

where P is a polynomial with integer coefficients.

We shall give a full detail in [6].

1. First, we define certain sequences and state some lemmata.

Definition 1. Let $u_n, v_n, (a)_n$ be sequences of numbers defined by

$$u_1=u_2=1, \qquad u_{n+2}=u_{n+1}+u_n,$$
$$v_1=1, \quad v_2=3, \quad v_{n+2}=v_{n+1}+v_n,$$
$$(a)_0=0, \quad (a)_1=1, \quad (a)_{n+2}=a\cdot(a)_{n+1}-(a)_n,$$

where a is a constant.

Lemma 1. (1) *If $m|n$, then $u_m|u_n$.*

(2) $2u_{m+n}=u_m v_n+u_n v_m$.

(3) $2v_{m+n}=5u_m u_n+v_m v_n$.

(4) $u_{m+n+1}=u_{m+1}u_{n+1}+u_m u_n$.

(5) $u_n v_n=u_{2n}$.

(6) $(u_n, v_n)=1$, if $3\nmid n$.

(7) $[(2x(2x)_n)_n/(2(2x)_n)_n]=x^n$.

Proof. For (1)∼(6), let $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$ and then we obtain $u_n=(\alpha^n-\beta^n)/\sqrt{5}$ and $v_n=\alpha^n+\beta^n$, from which the above formulae may be derived.

For (7), we put $p=(2x)_n$. By $(2x)_n>x^n$ we have $x^n(2p)_n \le (2xp)_n < (x^n+1)(2p)_n$.

Definition 2. We define sequences of numbers $|a|_n, \{a\}_n$ such that:

$$|a|_1=1, \quad |a|_2=a+1, \quad |a|_{n+2}=a\cdot|a|_{n+1}-|a|_n,$$
$$\{a\}_0=1, \quad \{a\}_1=a-1, \quad \{a\}_{n+2}=a\cdot\{a\}_{n+1}-\{a\}_n.$$