29. *A Characterization of Submodules of the Quotient Field of a Domain*

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1. Introduction. Let D be an elementary unique factorization domain with identity and K its quotient field. Let $\boldsymbol{P}$ be the set of the prime elements of D, and we consider the set $\boldsymbol{F}$ of the maps f from $\boldsymbol{P}$ into $\boldsymbol{Z}\cup\{-\infty\}$ (the set of integers and negative infinity), provided that there exists only a finite number of prime elements p such that $f(p)>0$ for each map f of $\boldsymbol{F}$. Let $M(f)$ be the set of the elements $x\in K$ with $V_p(x)\geq f(p)$ for all $p\in\boldsymbol{P}$, where V_p denotes the p-valuation of K. Then we can prove that $M(f)$ is a D-module, which is called an *f-module*. Now in [2], R. A. Beaumont and H. S. Zuckerman have characterized the additive groups of rational numbers. The purpose of this paper is to extend the results in [2] for an elementary unique factorization domain D and to investigate D-submodules of K related with f-modules.

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2. Properties of f-modules in an elementary unique factorization domain.

Let D be an elementary unique factorization domain (abv. EUFD) with the quotient field K, and let $\boldsymbol{P}$ be the set of all prime elements. Let a be a non-zero element of D and $a=\Pi_{j=1}^{s}p_j^{n_j}$ (n_j: positive integers) the factorization of a into prime factors. We define the valuation of K in the following way. We consider the map v_p of D into non-negative integers: $v_p(a)=n_j$, $v_p(0)=\infty$ for all p, and extend v_p to K as follows: $V_p(a)=v_p(ac)-v_p(c)$, where $0\neq a\in K$ and $ac\in D$ with $0\neq c\in D$. It is easy to see that the map V_p of K into integers does not depend on the choice of c, and satisfies the above conditions of the p-valuation. If $f(p)=0$, $f\in\boldsymbol{F}$, for all prime elements p, it is easily verified that $M(f)=D$.

Proposition 2.1. *Let D be EUFD with the quotient field K. Then $M(f)\supseteq M(f')$ if and only if $f(p)\leq f'(p)$ for each element p of $\boldsymbol{P}$.*

Proof. "If part" is evident. Suppose that $M(f)\supseteq M(f')$, and assume that $f(p_0)>f'(p_0)$ for some element p_0 of $\boldsymbol{P}$. Let $\boldsymbol{Q}=\{p_{k_1},\cdots,p_{k_r}\}$ be the set of the primes with $f(p_{k_j})>0$ or $f'(p_{k_j})>0$ ($j=1,\cdots,r$). If p_0 is in $\boldsymbol{Q}$, we take out it from the set, and if $f'(p_0)=-\infty$, we set $f'(p_0)=-n$ by taking an integer $n>0$ such that $f(p_0)>-n$. Let a