26. On Some Examples of Non-normal Operators. III

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1. Introduction. In the previous note [3 ; II], we have introduced the hen-spectra of operators. If T is an operator acting on a Hilbert space $\mathfrak{H}$ with the spectrum $\sigma(T)$, then the *hen-spectrum* $\tilde{\sigma}(T)$ is the complement of the unbounded component of $\sigma(T)^c$ where M^c is the complement of a set M in the complex plane. Clearly, the hen-spectrum is a compact set in the plane with the connected complement, and we have proved in [3 ; II, Proposition 2].

$$(1)\qquad \sigma(T)\subset\tilde{\sigma}(T)\subset \mathrm{co}\,\sigma(T)\subset \overline{W}(T),$$

where $\mathrm{co}\,M$ is the convex hull of M, $\overline{M}$ the closure of M, and $W(T)$ is the numerical range of T.

In the previous note [3 ; II], we are concerned with growth conditions: An operator T is called to satisfy the *condition* (G_1) (resp. (H_1)) if

$$(2)\qquad \|(T-\lambda)^{-1}\|\leqq\frac{1}{\mathrm{dist}\,(\lambda, X)}$$

for $\lambda\notin X$ and $X=\sigma(T)$ (resp. $X=\tilde{\sigma}(T)$). By (2), we have, $T\in(G_1)$ implies $T\in(H_1)$, and $T\in(H_1)$ implies that T is a convexoid in the sense of Halmos [5], i.e. $\overline{W}(T)=\mathrm{co}\,\sigma(T)$.

In the present note, we shall concern with spectral sets introduced by von Neumann: A closed set S in the complex plane called a *spectral set for an operator* T if

$$(3)\qquad \sigma(T)\subset S$$

and

$$(4)\qquad \|f(T)\|\leqq\|f\|_S,$$

where f is a rational function with poles off S and

$$\|f\|_S=\sup_{z\in S}|f(z)|,$$

cf. [6] for details. If S is a spectral set for T and $S\subset S'$, then S' is also a spectral set for T. A fundamental theorem for spectral set is

Theorem A (von Neumann). *The* (*closed*) *unit disk D is a spectral set for every contraction.*

The following theorem, also due to von Neumann, is a direct consequence of Theorem A:

Theorem B. $\{\alpha\,;|\alpha-\lambda|\geqq\beta\}$ *is a spectral set for T if and only if* $\|(T-\lambda)^{-1}\|\leqq 1/\beta$.