56. An Inequality for 4-Dimensional Kählerian Manifolds

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1. Introduction. Let (M, g, J) be a Kählerian manifold with almost complex structure J and Kählerian metric tensor g. By $R=(R^i_{jkl})$, $(R_{jk})=(R^r_{jkr})$, and S we denote the Riemannian curvature tensor, the Ricci curvature tensor, and the scalar curvature, respectively. By dM we denote the volume element of (M, g, J). By $\chi(M)$ we denote the Euler-Poincaré characteristic of M. By $\mathrm{Vol}(M)$ we denote the total volume of (M, g, J).

Main theorem. *Let (M, g, J) be a (real) 4-dimensional compact Kählerian manifold. Then the following inequality holds:*

$$\chi(M)\geq\frac{1}{96\pi^2}\left[\int S^2dM-6(2-\beta)\int[R_{ij}-(S/4)g_{ij}][R^{ij}-(S/4)g^{ij}]dM\right], \tag{1.1}$$

where β is an arbitrary constant <1. The equality holds if and only if (M, g, J) is of constant holomorphic sectional curvature.

Furthermore, if (M, g, J) is an Einstein space, then

$$96\pi^2\chi(M)\geq S^2\,\mathrm{Vol}(M) \tag{1.2}$$

holds. The equality holds, if and only if (M, g, J) is of constant holomorphic sectional curvature.

We give an outline of the proof. First we need to find out inequalities concerning $(R_{ijkl}R^{ijkl})$, $(R_{jk}R^{jk})$ and S^2, such that the equality implies constancy of holomorphic sectional curvature. For this purpose we give a new characterization of the Weyl's conformal curvature tensor in § 3, and in the next section we give a characterization of the Bochner curvature tensor. In this process we have the best inequality (4.14).

2. Preliminaries. Let (M, g) be a Riemannian manifold of dimension m. By ∇ we denote the Riemannian connection with respect to g. If $R_{ijkl}=k(g_{jk}g_{il}-g_{jl}g_{ik})$ holds on M (at x, resp.) for a real number k, (M, g) is said to be of constant curvature (at x, resp.). We put

$$A(g)=R_{ijkl}R^{ijkl}-(2/(m-1))R_{jk}R^{jk}, \tag{2.1}$$

$$B(g)=R_{jk}R^{jk}-(1/m)S^2. \tag{2.2}$$

Then $A(g)\geq 0$ holds; the equality holds on M (at x, resp.) if and only if (M, g) is of constant curvature (at x, resp.). $B(g)\geq 0$ holds; the equality on M is equivalent to the fact that (M, g) is an Einstein space (cf.