51. On Some Hyperbolic Equations with Operator Coefficients

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(Comm. by Kôsaku YOSIDA, M. J. A., April 12, 1973)

1. We consider first the class of singular Cauchy problems for $u^m \in C^2(E)$ on $[0, T]$.

$$u^m_{tt}+(2m+1)\coth t\, u^m_t+m(m+1)u^m=A^2u^m \tag{1.1}$$

$$u^m(0)=u_0;\qquad u^m_t(0)=0 \tag{1.2}$$

where $u_0 \in E$ is given and A is the generator of a locally equicontinuous group $T(t)=\exp At$ in a complete locally convex Hausdorff space E (cf. [9]). When A^2 is replaced by the Laplace-Beltrami operator Δ in function spaces E over $M=SL(2,R)/SO(2)$ and $m\geqslant 0$ is an integer, these equations arise in a canonical way from certain Lie group theoretic considerations and are parallel to the corresponding Euler-Poisson-Darboux (EPD) equations (cf. [2], [4], [10]); in fact there are many parallel theories for canonical classes of singular Cauchy problems but we will only deal here with (1.1)–(1.2) (cf. [5], [10] for other situations).

Now there are two canonical recursion relations arising from the group theory when A^2 is replaced by Δ which we write in the form

$$u^m_t+2m\coth t\, u^m=2m\,\mathrm{csch}\, t\, u^{m-1} \tag{1.3}$$

$$u^m_t=\frac{\sinh t}{2(m+1)}[A^2-m(m+1)]u^{m+1} \tag{1.4}$$

and (1.3) leads directly to a generalized Sonine formula (sh = sinh and ch = cosh)

$$\mathrm{sh}^{2m} t\, u^m(t)=c(m,l)\int_0^t(\mathrm{ch}\, t-\mathrm{ch}\, y)^{l-1}\,\mathrm{sh}^{2m-2l+1} y\, u^{m-l}(y)dy \tag{1.5}$$

where $c(m,l)=2^l\Gamma(m+1)/\Gamma(m-l+1)\Gamma(l)$ and (temporarily) $m\geqslant l\geqslant 1$ are integers. Thus, for example, when $m=l\geqslant 1$ is an integer one connects u^m to the mean value solution u^0 and (1.4)–(1.5) yield a growth theorem $u^m_t\geqslant 0$ for $m\geqslant 0$ whenevr $[\Delta-m(m+1)]u_0\geqslant 0$ (since $\Delta u^0(t,u_0)=u^0(t,\Delta u_0)$). This and similar convexity theorems (see [2]; [4]; [10]) are parallel to those of Weinstein [11] for EPD equations with $m\geqslant 0$ arbitrary (cf. also [1], [7], [12]). The Weinstein recursion relations for the EPD theory correspond to a version of (1.4) plus a relation connecting u^m to u^{-m} (see remarks after (3.1)); a parallel form of a version of (1.3) was also known. The (1.3)–(1.4) analogues were however first systematically exploited together in existence-uniqueness theory for EPD

*) Some of this work was done while the author was visiting at the University of Maryland.