70. *Finitely Generated $\mathfrak{N}$-Semigroup and Quotient Group*

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1. Introduction. An $\mathfrak{N}$-semigroup is a commutative cancellative archimedean semigroup which has no idempotent. The structure and construction of finitely generated or power joined $\mathfrak{N}$-semigroups were studied by [2], [3], [5], [6], and also by [4] from the more general point of view. This paper treats finitely generated $\mathfrak{N}$-semigroups as subsemigroups of the direct product of the positive integer semigroup and a finite abelian group by using the quotient group and its torsion subgroup. Finitely generated $\mathfrak{N}$-semigroups are characterized by their quotient group.

2. Preliminaries. In this paper we denote the additive semigroup of integers, positive integers, negative integers, non-negative integers, and positive rational numbers by Z, Z_+, Z_-, Z_+^0, and R respectively.

Proposition 1 ([1], [6]). *Let G be an abelian group and $I: G\times G \to Z_+^0$ be a function satisfying*

(1.1) $I(\alpha,\beta)=I(\beta,\alpha)$ *for all* $\alpha,\beta\in G$.

(1.2) $I(\alpha,\beta)+I(\alpha\beta,\gamma)=I(\alpha,\beta\gamma)+I(\beta,\gamma)$ *for all* $\alpha,\beta,\gamma\in G$.

(1.3) $I(\varepsilon,\alpha)=1$ (*ε being the identity of* G) *for all* $\alpha\in G$.

(1.4) *For each $\alpha\in G$ there is $m\in Z_+$ such that* $I(\alpha,\alpha^m)>0$.

Let $S=\{(x,\alpha): x\in Z_+^0, \alpha\in G\}$. Define an operation

$$(x,\alpha)(y,\beta)=(x+y+I(\alpha,\beta),\alpha\beta).$$

Then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup can be obtained in this manner.

S is denoted by $S=(G;I)$. The group G is termed the structure group of S with respect to $(0,\varepsilon)$, the function I is called an index function or $\mathcal{I}$-function corresponding to G. For a given $\mathfrak{N}$-semigroup S, for each $a\in S$, the relation ρ_a on S is defined by

$$x\,\rho_a\,y \text{ if and only if } a^m x=a^n y \text{ for some } m,n\in Z_+.$$

Then ρ_a is a congruence on S and $G_a=S/\rho_a$ is an abelian group. Each ρ_a-class contains exactly one element p_α, $\alpha\in G_a$, such that $p_\alpha\notin Sa$. Then S is isomorphic onto $(G_a;I_a)$ where $p_{\alpha\beta}=a^{I_a(\alpha,\beta)}p_\alpha p_\beta$.

A commutative semigroup S is called power joined if for every $a,b\in S$ there are $m,n\in Z_+$ such that $a^m=b^n$. If S is power joined, it is archimedean.

Proposition 2 ([5]). *An $\mathfrak{N}$-semigroup $S=(G;I)$ is power joined if*