97. Note on Generalized Atomic Sets of Formulas

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In his paper [2], H. J. Keisler introduced the concept of generalized atomic sets of formulas and made interesting investigations on the theory of models with generalized atomic sets. Recently, G. Grätzer posed the following problem ([1; Problem 71 in p. 299]): *Let F and G be generalized atomic sets. Under what conditions are the corresponding homomorphism and substructure concepts equivalent?* The purpose of this note is to give an answer to this problem. We shall actually find an answer to such a problem concerning generalized atomic sets in a wider sense.

§1. Preliminaries. Let L be a first order language with equality. A formula Φ of L which contains at most some of distinct variables $x_1, \cdots, x_n$ as free variables is denoted by $\Phi(x_1, \cdots, x_n)$ if the variables $x_1, \cdots, x_n$ need to be indicated. If $t_1, \cdots, t_n$ are terms of L, we denote by $\Phi[t_1, \cdots, t_n]$ the formula obtained from $\Phi(x_1, \cdots, x_n)$ by substituting all free occurrences of $x_1, \cdots, x_n$ by the terms $t_1, \cdots, t_n$ respectively. Let $\mathfrak{A}$ be a structure for L. The domain of $\mathfrak{A}$ is denoted by $D[\mathfrak{A}]$. Let $\Phi(x_1, \cdots, x_n)$ be any formula of L, and let $a_1, \cdots, a_n$ be any elements in $D[\mathfrak{A}]$. Then we write $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$, if $a_1, \cdots, a_n$ satisfy $\Phi(x_1, \cdots, x_n)$ in $\mathfrak{A}$ when the free variables $x_1, \cdots, x_n$ are assigned the values $a_1, \cdots, a_n$ respectively. If $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$ holds for any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, we say that Φ is valid in $\mathfrak{A}$, and denote it by $\mathfrak{A} \models \Phi$. If $\mathfrak{A} \models \Phi$ holds for every structure $\mathfrak{A}$ for L, we write $\models \Phi$. Two formulas Φ and Ψ are said to be equivalent if $\models \Phi \leftrightarrow \Psi$.

Let F be any set of formulas of L. For any subset $\mathscr{X}$ of the set $\{\wedge, \vee, \neg, \forall, \exists\}$, we denote by $\mathscr{X}F$ the set of all formulas that can be formed from the formulas in F by using only the connectives and quantifiers in $\mathscr{X}$. If $\mathscr{X}$ is a one-element set, e.g. $\mathscr{X} = \{\exists\}$, we use the briefer notation $\exists F$ in place of $\{\exists\}F$. We also abbreviate the sets $\{\wedge, \vee, \neg\}$ and $\{\wedge, \vee, \forall, \exists\}$ by the symbols $\mathscr{B}$ and $\mathscr{P}$ respectively. Moreover we denote by $[F]$ the set consisting of all formulas in F and a fixed identically false formula ϕ of L, and by $\mathscr{E}_L(F)$ or briefly $\mathscr{E}_L F$ the set of all formulas of L that are equivalent to some formulas in F.

A set F of formulas of L is said to be *generalized atomic*, if the following four conditions hold:

(1) If $\Phi(x_1, \cdots, x_n) \in F$ and y is a variable of L whose new