112. A Note on the Abstract Cauchy Problem in a Banach Space

By Nobuhiro SANEKATA
Department of Mathematics, Waseda University

(Comm. by Kinjirô KUNUGI, M. J. A., July 12, 1973)

§1. Introduction. This note is concerned with the abstract Cauchy problem for a linear operator A (with domain $D(A)$ and range $R(A)$) in a Banach space X. The problem considered here is to characterize the complete infinitesimal generator (or infinitesimal generator) of a semigroup of some class in terms of the abstract Cauchy problem. This problem was first treated by Hille and in [4], Phillips characterized the infinitesimal generator (simply i.g.) of a semigroup of class (C_0). His formulation of the abstract Cauchy problem (for a linear operator A) is as follows:

ACP—Given an element $x \in X$, fixed a function $u(t)=u(t;x)$ satisfying (i) $u(t)$ is strongly continuously differentiable in $t \geq 0$, (ii) $u(t) \in D(A)$ and $(d/dt)u(t)=Au(t)$ for each $t>0$ and (iii) $u(0;x)=x$.

A purpose of this note is to characterize the complete infinitesimal generator (c.i.g.) of a semigroup of class $(C_{(k)})$ in terms of ACP. But some properties of semigroups of class $(C_{(k)})$ $(k \geq 1)$ suggest the other formulation of the abstract Cauchy problem (see [3; p. 251]). For this sake, we introduced a less restrictive formulation:

WCP—Given an element $x \in X$, find a function $u(t)=u(t;x)$ satisfying (i′) $u(t)$ is strongly continuous in $t \geq 0$ and strongly continuously differentiable in $t>0$ and conditions (ii) and (iii) in ACP.

We shall call the X-valued function $u(t)$ satisfying (i) (or (i′)), (ii) and (iii) the solution of (APC; A, x) (or WCP; A, x)). In comparison with the solution of ACP, the behavior of the derivative of the solution of WCP has no restriction near $t=0$. Therefore, this formulation is called the weak Cauchy problem in [2] and is denoted by WCP in this note. However, the relationship between ACP and WCP when A has a nonvacuous resolvent set is described in Lemma 1.2.

Now, we state our result.

Theorem 1.1. *Let A be a closed linear operator with dense domain and nonvacuous resolvent set, and let k be a positive integer. Suppose that for each $x \in D(A^k)$ there is a unique solution $u(t;x)$ of* (WCP; A, x) *(or* (ACP; A, x)) *such that $u(t;x) \in D(A^k)$ for each $t>0$. Then A is the c.i.g. of a semigroup $\{T(t)\}_{t>0}$ of class $(C_{(k)})$ (or $(C_{(k-1)})$) such that $u(t;x)$*