108. *Multiplicative Excessive Measures of Branching Processes*

By Masao NAGASAWA
Tokyo Institute of Technology

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1. Given continuous time Galton-Watson processes (abbreviated as CGW), we will consider the existence of "multiplicative" excessive measures of CGW processes. The existence of such measures is of importance in the duality of branching Markov processes. As is seen in the following theorem, no multiplicative excessive measure exists in some cases.

Given $\mu \geqq 0, \{\mu^n, n=1,2,3,\cdots\}$ is said to be *multiplicative* excessive measure (abbreviated as M-excessive measure) if

(1) $$\sum_{n=1}^{\infty} \mu^n T_t(n,m) \leqq \mu^m, \quad t \geqq 0, \quad m=1,2,3,\cdots,$$

where $T_t(n,m)$ is the transition probability of CGW process. We will denote M-excessive measures as $\hat{\mu}$. As usual,

$$h(u)=\sum_{n=0}^{\infty} q_n u^n,$$

is the generating function of $\{q_n; n \geqq 0\}$.[1)] Then we have

Theorem. (i) *When CGW process is critical*[2)] *there exists the unique M-excessive measure* $\hat{1}=\{1,1,\cdots\}$ *if and only if* $q_0=q_2=1/2$;
(ii) *When supercritical, there exist M-excessive measures if and only if* $q_0 \leqq 1/2$. *In this case,* $0 \leqq \mu \leqq 1/2q_0$ *gives* $\hat{\mu}$;
(iii) *When subcritical, there exist M-excessive measures if and only if* $1/2 \leqq q_0 \leqq r/2$.[3)] *In this case,* $1/r \leqq \mu \leqq 1/2q_0$ *gives* $\hat{\mu}$.

The theorem is a consequence of the following

Lemma. *For* $\mu>0$, $\hat{\mu}$ *is an M-excessive measure if and only if*

(a) $$2q_0 \leqq \mu^{-1}, \text{ and}$$

(b) $$h(\mu^{-1}) \leqq \mu^{-1}.$$[4)]

Remark. It is easy to see that the condition (b) is equivalent to (b′) $q \leqq \mu^{-1} \leqq r$.

Example 1. When $q_0=0$, CGW process is supercritical, and $q=0$

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1) $q_n \geqq 0$, and $\sum_{n=0}^{\infty} q_n=1$.

2) We call CGW process is critical, supercritical, and subcritical, if $h'(1)=1$, $h'(1)>1$, and $h'(1)<1$, respectively.

3) $0 \leqq q \leqq r$ are the roots of $h(u)-u=0$.

4) To prove the lemma a characterization of excessive measures in terms of infinitesimal generator (or resolvents) provides a useful tool (cf. [2]).