106. Note on Potential Operators on L^p

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The purpose of the present paper is to prove in an abstract setting a theorem on the existence and non-existence in L^p $(1\leqslant p<\infty)$ of potential operators associated with a (temporally homogeneous) Markov process with an invariant measure. We shall apply this result to a consideration of abstract "semi-linear Poisson's equations" (cf. Konishi [10]) in L^1 and L^2.

Remember that an equi-continuous semi-group $\{T_t\}_{t\geqslant 0}$ of class (C_0) in a Banach space X is said to admit a *potential operator* V (in the sense of Yosida [19]) if its infinitesimal generator A admits a densely defined inverse A^{-1}: $V=-A^{-1}$ (see also Yosida [21] and Chapter XIII, 9 of Yosida [22]). We shall make use of the fact that $\{T_t\}_{t\geqslant 0}$ admits a potential operator if and only if $\lim_{\lambda\downarrow 0}\lambda(\lambda I-A)^{-1}f=0$ for every $f\in X$. (See also Theorem 2.2 of Sato [14] for several other criteria for the existence of potential operators.)

1. Potential operators on L^p. Let $\mathfrak{B}$ be a σ-additive family of subsets of a set $S\neq\phi$ and $P(t,x,E)$, $t>0$, $x\in S$, $E\in\mathfrak{B}$, be the transition probability of a Markov process on the phase space $(S,\mathfrak{B})$ with a (σ-finite) invariant positive measure m (see, e.g., Yosida [22], XIII, 1). Then by the relation:

$$(T_{p,t}f)(x)=\int_S P(t,x,dy)f(y),\qquad f\in L^p\equiv L^p(S,\mathfrak{B},m),$$

a non-negative contraction semi-group $\{T_{p,t}\}_{t>0}$ in real L^p is defined for each $1\leqslant p\leqslant\infty$. Let $\mathscr{B}$ be a closed subspace of L^∞ such that

(1) $\qquad\mathscr{B}\cap L^1$ is dense in L^p, $\qquad 1<p<\infty$,

and that $T_{\infty,t}$, $t>0$, leaves $\mathscr{B}$ invariant. Denote by $T_{\mathscr{B},t}$, $t>0$, the restriction of $T_{\infty,t}$ to $\mathscr{B}$. We assume that the semi-group

(2) $\qquad\{T_{\mathscr{B},t}\}_{t\geqslant 0}\subset L(\mathscr{B},\mathscr{B})$ is of class (C_0)

and, moreover, that the semi-groups

(3) $\qquad\{T_{p,t}\}_{t\geqslant 0}\subset L(L^p,L^p)$, $1\leqslant p<\infty$, are of class (C_0).

We denote by $A_{\mathscr{B}}$ and A_p's their infinitesimal generators respectively.

Theorem. *Suppose that the semi-group*

(4) $\qquad\{T_{\mathscr{B},t}\}_{t\geqslant 0}$ *admits a potential operator.*

Then we have the following:

(i) *The semi-group $\{T_{p,t}\}_{t\geqslant 0}$ in L^p, $1<p<\infty$, admits a potential operator.*