150. *On the Character Rings of Finite Groups*

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Introduction. Let G be a finite group. In this paper all groups are finite and all characters are assumed to be characters of representations over the complex field. As is well known, every character of G is the sum of irreducible characters of G and the set of characters of G is closed under addition and multiplication. It is often convenient to consider also the difference of two characters (see [1, Chapter 6]). From this fact we shall be concerned with the ring generated by the irreducible characters χ_k of G over the ring Z of rational integers. The ring thus obtained we denote by $R(G)$, and call it the character ring of G. In this paper we deal with this character ring $R(G)$.

Clearly, $R(G)$ is a commutative Z-algebra. Its unity element is the principal character of G. Moreover every element of $R(G)$ is uniquely expressible as a Z-linear combination of the χ_k. If G is abelian, it is known that $R(G)$ is isomorphic to the group ring ZG (see e.g. [5] or [6]). However, in general, it is difficult to give a characterization of character rings. On the other hand, it is possible to state a little further the structure of the ring $Q \otimes_Z R(G)$, where Q denotes the rational field. We note that the character ring $R(G)$ has non-zero nilpotents. This implies that the ring $Q \otimes_Z R(G)$ is semi-simple (cf. [3], [4]). Therefore $Q \otimes_Z R(G)$ is isomorphic to a direct sum of a finite number of fields K_i. In [6], Thompson showed this fact using the decomposition of unity element into a sum of orthogonal primitive idempotents. On the basis of these results we obtain some properties of the ring $Q \otimes_Z R(G)$.

In the first section of this paper we observe prime ideals of $R(G)$ and determine the minimal prime ideals. Next we discuss the structure of the field K_i This argument leads to the result that $Q \otimes_Z R(G)$ is determined by a permutation group on the set of conjugate classes of G. In particular, if G is a p-group, where p is an odd prime integer, then there is the set of integers which determines the ring $Q \otimes_Z R(G)$.

§ 1. Prime ideals of the character ring $R(G)$.

Suppose m is a multiple of the exponent of G. Let ε_m be a primitive m-th root of 1 over Q, and A the integral closure of Z in the cyclotomic field $F_m = Q(\varepsilon_m)$. Let $Cl(G)$ denote the set of all conjugate