148. On Normalizers of Simple Ring Extensions

By Shigeru TAKAMATSU and Hisao TOMINAGA
Department of Mathematics, Okayama University

(Comm. by Kenjiro SHODA, M. J. A., Nov. 12, 1973)

Throughout the present note, A will represent an (Artinian) simple ring with the center C, and B a regular subring of A with the center Z. Let V be the centralizer $V_A(B)$ of B in A, and N the normalizer $N_A(B)=\{a\in A^\cdot \mid B\tilde{a}=B\}$ of B in A. As is well-known, $B_0=BV=B\otimes_Z V$ is two-sided simple. Obviously, $N\subseteq N_A(V)$ and $B^\cdot V^\cdot$ is a normal subgroup of N. We fix here a complete representative system $\{u_\lambda \mid \lambda\in\Lambda\}$ of N modulo $B^\cdot V^\cdot$. As to notations and terminologies used without mention, we follow [2].

In case $A\neq(GF(2))_2$, it is known that if $N=A^\cdot$ then either $B=A$ or $B\subseteq C$ (see for instance [2; Proposition 8.10 (a)]). In what follows, we shall prove further results concerning N such as P. Van Praag [1] obtained for division ring extensions.

Lemma. *The ring $BN=\sum_{u\in N} Bu$ is a completely reducible B-B-module with homogeneous components $B_0u_\lambda(\lambda\in\Lambda)$. Furthermore, every irreducible B_0-B_0-module B_0u_λ is not isomorphic to B_0u_μ for $\mu\neq\lambda$.*

Proof. It is obvious that every $Bu(u\in N)$ is B-B-irreducible. Now, assume that Bu is B-B-isomorphic to Bu_λ and $u\leftrightarrow bu_\lambda(b\in B)$. Since $Bb=B$, b is a unit of B. For every $b'\in B$, we have $ub'\leftrightarrow bu_\lambda b'=b\cdot b'\tilde{u}_\lambda\cdot u_\lambda$ and $b'\tilde{u}\cdot u\leftrightarrow b'\tilde{u}\cdot bu_\lambda$, and so $b\cdot b'\tilde{u}_\lambda=b'\tilde{u}\cdot b$, whence it follows $B\mid \widetilde{bu_\lambda}=B\mid\tilde{u}$. Hence, we obtain $(bu_\lambda)^{-1}u\in V^\cdot$, which implies that $u\in B^\cdot V^\cdot u_\lambda$. Conversely, every $Bvu_\lambda(v\in V^\cdot)$ is B-B-isomorphic to Bu_λ, and hence we have seen that $\bigoplus_{\lambda\in\Lambda} B_0u_\lambda$ is the idealistic decomposition of the B-B-module BN. Finally, if B_0u_λ is B_0-B_0-isomorphic to B_0u_μ $(\mu\neq\lambda)$ then they are B-B-isomorphic, which yields a contradiction.

Corollary. *If $V\subseteq B$ then BN is the direct sum of non-isomorphic irreducible B-B-submodules, and conversely.*

Proposition 1. *Assume that $BN=A$.*

(1) $[A:B]_L=[A:B]_R=(N:B^\cdot V^\cdot)[V:Z]$.

(2) *If N' is a subgroup of N containing $B^\cdot V^\cdot$ then $BN'\cap N=N'$.*

(3) *If A' is a simple intermediate ring of A/B_0 then $A'=BN_{A'}(B)$.*

(4) *V/C is Galois.*

Proof. (1) is clear by Lemma.

(2) By Lemma, $BN'=\bigoplus_{\lambda\in\Lambda'} B_0u_\lambda$ with a suitable subset Λ' of Λ.