168. The Extension of Darboux's Method to Systems in Involution of Partial Differential Equations of Arbitrary Order in Two Independent Variables

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0. Introduction. Darboux's method of integrating a single partial differential equation of the second order in two independent variables is such that for those equations to which the method may be successfully applied, the solution of Cauchy's problem can be reduced to the integration of a system of ordinary differential equations (cf. E. Goursat [8], A. R. Forsyth [7]). The main aim of our investigation is the extension of Darboux's method to systems in involution of partial differential equations of arbitrary order with one unknown function of two independent variables by applying the theory of differential systems due to E. Cartan (E. Cartan [1]–[5], E. Goursat [9]).

Darboux's method is summarized, from our standpoint, as follows. Consider a single differential equation with an unknown function $z(x, y)$ of two independent variables x, y

$$F(x, y, z, p, q, r, s, t)=0, \tag{1}$$

where $p=\partial z/\partial x$, $q=\partial z/\partial y$, $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x\partial y$, $t=\partial^2 z/\partial y^2$.

The differential equation (1) can be represented by the differential system

$$\begin{cases} F=0, \quad dF=0, \quad \varpi=dz-pdx-qdy=0, \\ \varpi_1=dp-rdx-sdy=0, \quad \varpi_2=dq-sdx-tdy=0. \end{cases} \tag{2}$$

If $u(x, y, z, p, q, r, s, t)$ is an integral, independent of F, of one of the characteristic systems of the equation (1), then the differential system

$$F=0, \quad u=0, \quad dF=0, \quad du=0, \quad \varpi=0, \quad \varpi_1=0, \quad \varpi_2=0$$

has one-dimensional (Cauchy's) characteristics. Therefore if there exists one more integral of order two, independent of F and u, of the same characteristic system, then Cauchy's problem can be solved by integrating a system of ordinary differential equations. When neither characteristic systems of (1) have two independent integrals of order two, we prolong the system (2). The similar argument implies that, if one of the characteristic systems has two independent integrals (invariants) of possibly higher order, then we can solve Cauchy's problem by integrating a system of ordinary differential equations. This argu-