# 12. A Characterization of Nonstandard Real Fields

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Throughout this note, $(R, 0, 1, +, \cdot, \leq)$, or simply $R$, denotes the ordered field of real numbers, and $\hat{R}$ the union of all sets $R_n$ defined inductively by $R_0 = R$ and $R_{n+1} = \mathscr{P}(\bigcup_{i=0}^{n} R_i)$ $(n = 0, 1, 2, \cdots)$, where $\mathscr{P}(X)$ denotes the power set of $X$. Let $\mathcal{U}$ be a $\delta$-incomplete ultrafilter on an infinite set $I$. A *nonstandard real number* is defined to be an individual of the ultrapower of $\hat{R}$ with respect to $\mathcal{U}$, and the set $^*R$ of all nonstandard real numbers to be the value at $R_0$ of the mapping $a \mapsto {}^*a$ of $\hat{R}$ into $\hat{R}^I$ defined by $^*a(t) = a$ for all $t \in I$, where $=$ and $\in$ in $\hat{R}^I$ are defined for $\boldsymbol{a}, \boldsymbol{b} \in \hat{R}^I$ as follows: $\boldsymbol{a} = \boldsymbol{b}$ if and only if $\{t \in I : \boldsymbol{a}(t) = \boldsymbol{b}(t)\} \in \mathcal{U}$, and $\boldsymbol{a} \in \boldsymbol{b}$ if and only if $\{t \in I : \boldsymbol{a}(t) \in \boldsymbol{b}(t)\} \in \mathcal{U}$. Then as is known[*], $(^*R, {}^*0, {}^*1, {}^*+, {}^*\cdot, {}^*\leq)$ is a totally ordered field which will be referred in this note as the *$\mathcal{U}$-nonstandard real field*. Let $I$ be a set. By *nonstandard real field over I* we mean a totally ordered field which is isomorphic to some $\mathcal{U}$-nonstandard real field for a $\delta$-incomplete ultrafilter $\mathcal{U}$ on $I$.

The purpose of this note is to state a condition characterizing nonstandard real fields among totally ordered fields.

**Theorem 1.** *A totally ordered field $K$ is a nonstandard real field over a set $I$ if and only if it is non-Archimedean and is a homomorphic image of $R^I$, the ring of all real valued functions on $I$ with the pointwise addition and the pointwise multiplication.*

This result offers of course an axiom system for a nonstandard real field: *A nonstandard real field over a set $I$ is defined to be any non-Archimedean totally ordered field $K$ containing a complete Archimedean subfield $R_0$ such that $K$ is a homomorphic image of the ring $R_0^I$.*

Let $K$ be a totally ordered field. An element $x$ of $K$ is said to be *infinitely large* if $a < x$ for every rational element $a \in K$. Let $I$ be a set. For each real number $a$, let $^*a$ denote the constant mapping on $I$ defined by $^*a(t) = a$ for all $t \in I$. The ordering $\leq$ on the ring $R^I$ is defined as follows: $\boldsymbol{a} \leq \boldsymbol{b}$ if and only if $\boldsymbol{a}(t) \leq \boldsymbol{b}(t)$ for all $t \in I$.

**Proof of Theorem 1.** It suffices to prove the "if" part. Let $\varphi$ be the homomorphism of the ring $R^I$ onto $K$, that is, $\varphi$ is a mapping of $R^I$ onto $K$ such that $\varphi(\boldsymbol{a} + \boldsymbol{b}) = \varphi(\boldsymbol{a}) + \varphi(\boldsymbol{b})$ and $\varphi(\boldsymbol{ab}) = \varphi(\boldsymbol{a})\varphi(\boldsymbol{b})$ for all

[*] See for example, W. A. J. Luxemburg: What is nonstandard analysis. Amer. Math. Monthly, 80, 38–67 (1973).