6. On the Zeros of Heck's L-Functions

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1. Introduction. In this note we shall derive a density estimate near $\sigma=1$ of all L-functions with character defined mod $\tilde{\mathfrak{f}}$, where $\mathfrak{f}$ is an integral ideal of K, which is the extension of the rational field of degree n. Of course, such result has been already obtained in Fogels [1], and Gallagher [2] has extended his result to a large sieve type in the case $K=\boldsymbol{Q}$. He also says that Fogels' proof may be simplified by using his method in combination with the Brun-Titchmarsh inequality and Turan's power-sum method. His assertion is true even for the case of an arbitrary algebraic number field K. It is the main purpose of this note to offer to give its proof with some explicit constants. Our main tools are the same as what Gallagher has mentioned. But some of the theorems, which are well-known, are re-proved with some explicit constants. The proof will be a little more complicated than Gallagher's because we intend to get as small constants as possible. The author wishes to express his thanks to Prof. T. Tatuzawa, who encouraged him and gave him fruitful suggestions during preparing this note.

2. Notation and terminology. Let K be an algebraic number field of degree n with r_1 real conjugates and $2r_2$ complex conjugates, $\mathfrak{f}$ an integral ideal of K and $\tilde{\mathfrak{f}}$ denote a product of $\mathfrak{f}$ and q infinite prime spots of K $(0\leqq q\leqq r_1)$. Furthermore χ is a character defined mod $\tilde{\mathfrak{f}}$ and $\sum_{\bmod \tilde{\mathfrak{f}}}$ means the sum taken over all χ defined mod $\tilde{\mathfrak{f}}$. Functions $\mu(\mathfrak{a})$, $\varphi(\mathfrak{a})$, $\Lambda(\mathfrak{a})$ and $\psi_K(x)$ are generalizations of the Möbius, Euler, von Mangoldt and Chebychev function to K, respectively. All German letters will denote integral ideals of K, especially $\mathfrak{p}$ prime ideals. All estimates in $\ll$ or O are independent to $\mathfrak{f}$ and χ. All Latin letters will denote some constants.

3. Preliminary from the theory of functions and numbers. To begin with, we quote the functional equation of $L(s,\chi)$, where χ is primitive.

Theorem A. *We assume that χ is primitive. Let put*

$$A(\mathfrak{f})=2^{-r_2}\pi^{-n/2}\sqrt{|d|\,N(\mathfrak{f})},$$

where d is a discriminant of K, and