24. On 3-dimensional Interaction Information

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(Comm. by Kôsaku YOSIDA, M. J. A., Feb. 12, 1974)

1. Introduction. A notion of interaction information was introduced by McGill [2] and it was used in multivariate information analysis. Lately, it is shown that it plays an important role especially in the variables connected with Markov dependence [5]. However, it seems that the essential problem suggested by McGill: *under what conditions does 3-dimensional interaction information take positive, zero and negative values?* is not yet solved. The purpose of the present note is to show that it deeply relates to the trace of the product matrix of the three transition matrices, each of which represents the relationship between the variables. It is also shown that if the trace is equal to 1, the variables having zero interaction information constitute some intermediate dependence lying between Markovian dependence and independence. In addition, we show some processes realizing positive and negative interactions.

2. Definitions and basic properties. Consider the random variables X, Y and Z, taking only finite number of states $\{a_1, a_2, \cdots, a_L\}$, $\{b_1, b_2, \cdots, b_M\}$ and $\{c_1, c_2, \cdots, c_N\}$, respectively, where L, M and N are positive integers. $P(XYZ)$ denotes the function taking the joint probability value $p(ijk)=P(X=a_i, Y=b_j, Z=c_k)$ when $X=a_i, Y=b_j$ and $Z=c_k$. Then, 3-dimensional interaction information of the variables X, Y and Z is defined by

$$J=J(XYZ)=E\{\log [P(XYZ)P(X)P(Y)P(Z) \\ /(P(XY)P(YZ)P(ZX))]\} \tag{2.1}$$

or, equivalently, using the conditional probabilities,

$$J=E\{\log [P(XYZ)/P(Y\,|\,X)P(Z\,|\,Y)P(X\,|\,Z))]\} \tag{2.2}$$

where E means the expectation over all $P(XYZ)$. This may be more instructive if we rewrite it as

$$\begin{aligned} J&=E\{\log [P(XZ\,|\,Y)/(P(X\,|\,Y)P(Z\,|\,Y))]\} \\ &\quad -E\{\log [P(XZ)/(P(X)P(Z))]\} \\ &=EI(X,Z\,|\,Y)-I(X,Z). \end{aligned} \tag{2.3}$$

The first term of (2.3) is the conditional information between X and Z, given Y, and the second, the mutual information between X and Z. Thus, J may be considered as a measure which suggests the effect of Y with respect to X and Z. Since J is symmetric with respect to each variable (cf. [5]), these representations do not lose the generality.