21. *On Linear Operators with Closed Range*

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(Comm. by Kôsaku YOSIDA, M. J. A., Feb. 12, 1974)

Several conditions for a linear operator to have closed range are known (*e.g.* Browder [2], Baker [1], *etc.*). In this short report we will find another condition which is different from those due to Banach, Browder, Baker, *etc.* and will prove to be useful from practical point of view if we apply this theorem to the theory of boundary problems for linear differential equations (*cf.* [4], [5]). A special case was studied by Tréves [7], page 51, where spaces are Fréchet and operators become epimorphisms. Our proof depends on Pták's proof of the open mapping theorem.

Let E and F be (Hausdorff) locally convex spaces, and T a densely defined closed linear operator of E into F. Let E' denote the dual space of E. Let ${}^tT$ be the dual operator of T and $D({}^tT)$ its domain. We call $\mathfrak{B}$ a basis of continuous seminorms on E if its elements are continuous seminorms on E and for every continuous seminorm p on E there exist $q \in \mathfrak{B}$ and a positive constant C such that $p(x) \leqq C \cdot q(x)$, $x \in E$. For $x' \in E'$ and a continuous seminorm p on E, we write

$$\|x'\|_p = \inf\{C>0\,;\, |x'(x)| \leqq C \cdot p(x),\, x \in E\}.$$

If there exists no such positive constant C, we set $\|x'\|_p = \infty$.

Recall that T is called almost open if, for each neighborhood U of $0 \in E$, the closure of $T(U)$ in F is a neighborhood of $0 \in F$. A locally convex space E is said to be B-complete if a linear continuous and almost open mapping of E onto any locally convex space F is open. Fréchet spaces and strong duals of Fréchet spaces are B-complete (*cf.* [6]).

Theorem. *Let E be a B-complete space and F a fully barrelled space, that is, every closed subspace of F is barrelled. Let $\mathfrak{B}_E$ and $\mathfrak{B}_F$ be bases of continuous seminorms on E and F respectively. Then the range of T is closed if and only if the following two conditions are satisfied.*

(1) *For every seminorm $p \in \mathfrak{B}_E$ there exists another seminorm $q \in \mathfrak{B}_F$ such that $y' \in D({}^tT)$ and $\|{}^tT(y')\|_p < \infty$ implies the existence of $z' \in D({}^tT)$, which satisfies ${}^tT(y') = {}^tT(z')$ and $z' = 0$ on the null space of q.*

(2) *For seminorms $p \in \mathfrak{B}_E$ and $q \in \mathfrak{B}_F$ there exist another seminorm $r \in \mathfrak{B}_E$ and a positive constant C such that the following holds. For every $y' \in D({}^tT)$, which is equal to zero on the null space of q, there*