48. Approximate Solutions for Some Non-linear Volterra Integral Equations

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In this short note we give generalized ε-approximate solutions $x(t;\xi,\varepsilon)$ of the following non-linear integral equations of Volterra-type

$$\text{(P)} \qquad x(t)=f(t)+\int_0^t g(t,s,x(s))ds.$$

Under very general assumptions on $f(t)$ and $g(t,s,x)$ similar to the Carathèodory-type, R. K. Miller and G. R. Sell [1] proved the local existence theorem by applying the fixed point theorem of Schauder-Tychonoff. We shall prove that their assumptions in [1] assure the existence of generalized ε-approximate solutions $x(t;\xi,\varepsilon)$ of (P) and give some interesting properties of $x(t;\xi,\varepsilon)$ which will play an essential role in our sequel paper [3]. As an easy application of our results, we can show another existence proof of a solution of (P).

Let $|x|$ denote the Euclidean norm of a vector x of R^n. For each interval I containing O and each subset K of R^n, we define a space $\mathcal{C}(I;k)$ by the set of all continuous functions with domain I and range in K with the compact-open topology. Then $\mathcal{C}[0,\alpha]=\mathcal{C}([0,\alpha];R^n)$ is the Banach space of continuous functions on $[0,\alpha]$ with the norm of uniform convergence. We note that the space $\mathcal{C}[0,\alpha)=\mathcal{C}([0,\alpha);R^n)$ is not a Banach space but a Frèchet space. Denote by $\mathcal{L}^1[0,\alpha]$ the Banach space consisting of all Lebesgue measurable functions $x:[0,\alpha]\to R^n$ with finite norm $\int_0^\alpha |x(t)|\,dt<\infty$.

We assume the following hypotheses which are somewhat weaker than those in [1].

(H1) The function f is defined and continuous for all t in $R^+=\{t\in R: t\geqq 0\}$ with values in R^n.

(H2) Let $g(t,s,x)$ be a function defined on $R^+\times R^+\times R^n$ with values in R^n such that

(i) for each fixed $(t,x)\in R^+\times R^n$, $g(t,s,x)$ is Lebesgue measurable in s and $g(t,s,x)=0$ for $s>t$, and

(ii) for each fixed $(t,s)\in R^+\times R^+$ such that $s\leqq t$, $g(t,s,x)$ is continuous in x.

(H3) For each real number $l>0$ and each compact subset K of R^n, there exists a function $m(t,\cdot)\in\mathcal{L}^1[0,t]$ for each $t\in[0,l]$ such that