45. A Note on Hypercommutativity of Operators in Real Banach Spaces

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In [1], C. Apostol generalized to a complex Banach space an invariant subspace theorem of C. Pearcy and N. Salinas in a complex Hilbert space [8]. In a finite dimensional complex vector space every linear operator has at least one eigenvector (one-dimensional invariant subspace). This result which played a fundamental role in the development of the theory of complex vector spaces does not apply in the case of real spaces. The purpose of this note is to show the corresponding assertion of [1] in real Banach spaces. We base our arguments on C. Apostol's paper [1].

In this note, X will denote a separable Banach space over R (the set of all real numbers) of dimension greater than two, $\mathcal{B}(X)$ the algebra of all bounded linear operators acting in X, $\mathcal{E}(X)$ the set of all finite dimensional subspaces of X. If M is a non-empty subset of X and $x \in X$, the *distance* from x to M, $d(x, M)$, is defined by $d(x, M) = \inf \{\|x-y\| : y \in M\}$. In the sequel, a *subspace* means a closed linear manifold.

Definition 1 ([5]). Given a sequence $\{X_n\}$ of subspaces of X, define $\liminf X_n$ to be

$$\liminf X_n = \{x \in X : \lim d(x, X_n) = 0\}.$$

It is clear that $\liminf X_n$ is a subspace of X and $\liminf X_{n_k} = \liminf X_n$ for any subsequence $\{n_k\}$ of $\{n\}$. If for every $n \geqq 1$, X_n is a subspace of Y_n, then $\liminf X_n \subset \liminf Y_n$.

Definition 2. Let $\mathcal{A}$ be a set of operators, $\mathcal{A} \subset \mathcal{B}(X)$. Then an *invariant subspace for* $\mathcal{A}$ is an invariant subspace for all operators in $\mathcal{A}$.

Definition 3. Let $\{X_n\} \subset \mathcal{E}(X)$ and $\mathcal{C}_n \subset \mathcal{B}(X_n)$. We define $\liminf \mathcal{C}_n$ to be

$$\liminf \mathcal{C}_n = \left\{T \in \mathcal{B}(X) : \lim \left(\inf_{S_n \in \mathcal{C}_n} \|T | X_n - S_n\| \right) = 0 \right\}.$$

It is clear that $\liminf \mathcal{C}_{n_k} = \liminf \mathcal{C}_n$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\mathcal{C}_n, \mathcal{D}_n \subset \mathcal{B}(X_n)$. If for every $n \geqq 1$, $\mathcal{C}_n \subset \mathcal{D}_n$, then $\liminf \mathcal{C}_n \subset \liminf \mathcal{D}_n$.

Lemma 1 (P. Meyer-Nieberg [6] or [5]). *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be two*