71. A Note on Nonlinear Differential Equation in a Banach Space

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1. Let E be a Banach space with the dual space E^*. The norms in E and E^* are denoted by $\| \|$. We denote by $S(u, r)$ the closed sphere of center u with radius r.

It is our object in this note to give a sufficient condition for the existence of the unique solution to the Cauchy problem of the form

(1.1) $$u'(t)=f(t, u(t)), \qquad u(0)=u_0 \in E,$$

where f is a E-valued mapping defined on $[0, T]\times S(u_0, r)$.

We compare the differential equation (1.1) with the scalar equation

(1.2) $$w'(t)=g(t, w(t)),$$

where $g(t, w)$ is a function defined on $(0, a]\times[0, b]$ which is measurable in t for fixed w, and continuous monotone nondecreasing in w for fixed t. We say w is a solution of (1.2) on an interval I contained in $[0, a]$ if w is absolutely continuous on I and if $w'(t)=g(t, w(t))$ for a.e. $t \in I^\circ$, where I° is the set of all interior points of I.

We assume that g satisfies the following conditions:

(i) There exists a function m defined on $(0, a)$ such that $g(t, w) \leqq m(t)$ for $(t, w) \in (0, a)\times[0, b]$ and for which m is Lebesgue integrable on (ε, a) for every $\varepsilon>0$.

(ii) For each $t_0 \in (0, a]$, $w\equiv 0$ is the only solution of the equation (1.2) on $[0, t_0]$ satisfying the conditions that $w(0)=(D^+w)(0)=0$, where D^+w denotes the right-sided derivative of w.

2. Let g be as in Section 1. Then we have the following lemmas.

Lemma 2.1. *Let $\{w_n\}$ be a sequence of functions from $[0, a]$ to $[0, b]$ converging pointwise on $[0, a]$ to a function w_0. Let $M>0$ such that $|w_n(t)-w_n(s)|\leqq M|t-s|$ for $s, t \in [0, a]$ and $n\geqq 1$. Suppose further that for each $n\geqq 1$*

$$w_n'(t)\leqq g(t, w_n(t)) \qquad \textit{for } t \in (0, a)$$

such that $w_n'(t)$ exists. Then w_0 is a solution of (1.2) *on* $[0, a]$.

For a proof see [4].

Lemma 2.2. *Let $M>0$ and let $\{w_n\}$ be a sequence of functions from $[0, a]$ to $[0, b]$ with the property that $|w_n(t)-w_n(s)|\leqq M|t-s|$ for all $s, t \in [0, a]$ and $n\geqq I$. Let $w=\sup_{n\geqq 1} w_n$, and suppose that $w_n'(t) \leqq g(t, w_n(t))$ for $t \in (0, a)$ such that $w_n'(t)$ exists. Then w is a solution of* (1.2) *on* $[0, a]$.