70. Borel Structure in Topological $*$-algebras and Their Duals

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1. Introduction. One of the useful tools for studying the structure of a locally compact group or Banach $*$-algebra A is the *dual space* $\hat{A}$ of all its equivalence classes of irreducible representations in Hilbert space. In this paper, we deal with the Borel structure of a dual space for a topological $*$-algebra. It will be shown that the dual space $\hat{\mathcal{D}}(G)$ of the topological $*$-algebra $\mathcal{D}(G)$, where G is a σ-compact Lie group, coinsides with the dual space $\hat{G}$ of the σ-compact Lie group G and that if in addition G satisfies some conditions the dual space $\hat{\mathcal{D}}(G)$ is an analytic Borel space.

From these results, we shall conclude that a connected semi-simple Lie group and a connected nilpotent Lie group are type 1.

For locally convex spaces and their related notions, see [6] and for Borel structures and their related notions, see [4]. The proofs are omitted, and the details will appear elsewhere. The author would like to express his thanks Prof. O. Takenouchi for his helpful comments.

2. Topological $*$ algebra. A *topological algebra* is an algebra and a topological vector space over the complex number field such that ring multiplication $\circ$ is jointly continuous. A topological algebra E with a mapping $*$ of E into itself is called a *topological $*$-algebra* if the following conditions are satisfied: (1) $(x^*)=x$, (2) $(x\circ y)^*=x^*\circ y^*$, (3) $(x+y)^*=x^*+y^*$, (4) $(\lambda x)^*=\bar{\lambda}x^*$ for every $x, y \in E$ and scalar λ. By a *representation*, we mean a mapping T of E into $\mathcal{L}(H, H)$, the set of all continuous linear mapping of a Hilbert space H into itself, which satisfies the following conditions: (1) $T(x+y)=T(x)+T(y)$, (2) $T(\lambda x)=\lambda T(x)$, (3) $T(x\circ y)=T(x)T(y)$, (4) $T(x^*)=T(x)^*$ for every $x, y \in E$ and scalar λ. A representation is said to be *cyclic* if there exists an element h_0 (which is called a *cyclic element* for T) in the Hilbert space H such that the set $\{T(x)h_0 \mid x \in E\}$ is dense in H. The continuity, the irreducibility and the equivalency are defined similarly to the case of the unitary representations of a topological group. A unitary representation U, of a topological group G in a Hilbert space H, is said to be *continuous* at g_0 if $U(g)h \to U(g_0)h$ as $g \to g_0$ in G for every $h \in H$.

In what follows by a representation, we shall mean a continuous representation.