68. A Theorem on Riemannian Manifolds of Positive Curvature Operator

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Let M^n $(n>2)$ be a compact orientable Riemannian manifold. If there exists a positive constant k such that

$$(*) \qquad -R_{hjil}u^{hj}u^{il} \geqq 2ku_{ij}u^{ij}$$

holds good for any skew symmetric tensor u_{ij} at any point, then M^n is called to be of positive curvature operator. M. Berger [1] has proved $b_2(M)=0$ for the second Betti number of such manifolds, and then $b_i(M)=0$ by D. Meyer [3] for $i=1,\cdots,n-1$.

The purpose of this note is to prove the following.

Theorem. *If a compact orientable Riemannian manifold M^n $(n>2)$ of positive curvature operator satisfies*

$$(\sharp) \qquad \nabla^h R_{hjil}=0,$$

then M^n is a space of constant curvature.

We remark that the condition ($\sharp$) is satisfied when M^n has one of the following properties:

(i) the Ricci tensor is proportional to the metric tensor,

(ii) the Ricci tensor is parallel,

(iii) conformally flat, and the scalar curvature is constant.

Denoting the Ricci tensor by $R_{ji}=R_{hji}{}^h$ we define a scalar function K by

$$K=R_{lm}R^{ljih}R^m{}_{jih}+(1/2)R^{lmpq}R_{lmjh}R^{jh}{}_{pq}+2R^{ljmh}R_{lpmq}R^p{}_j{}^q{}_h.$$

Then we have

Lemma 1 ([2], [4]). *In a compact orientable Riemannian manifold, the integral formula*

$$\int_M \left\{K-|\nabla^h R_{hjil}|^2\right\}d\sigma=-\frac{1}{2}\int_M |\nabla_p R_{hjil}|^2\, d\sigma$$

holds good, where $|A_{jih}|^2=A_{hji}A^{hji}$, etc.

As it follows from ($\sharp$) that

$$\int_M K d\sigma=-\frac{1}{2}\int_M |\nabla_p R_{hjil}|^2\, d\sigma \leqq 0,$$

we shall calculate K under the condition ($*$).

Let P be any point of M^n and consider all quantities with respect to an orthonormal base field around P. For fixed k, j, i, h we define a local skew symmetric tensor field $u_{lm}^{(kjih)}$ by