

## 59. On a Problem of Fossum

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Robert M. Fossum proposes the following problem in his book "The divisor class group of a Krull domain":\*)

**Problem.** Let  $k$  be a field of characteristic not equal to 2, and  $F(X_1, X_2, X_3, X_4)$  a non-degenerate quadratic form over  $k$ . Find a necessary and sufficient condition in order that  $A_F = K[X_1, X_2, X_3, X_4]/(F)$  may be a factorial ring.

The purpose of the present note is to give the answer of the problem. In this note, we employ the same terminology and notation as of [F].

**Lemma 1.** Let  $F = X_1^2 + aX_2^2 + bX_3^2 + cX_4^2$  ( $abc \neq 0, a, b, c \in k$ ). If  $c = ab$  then  $\text{Cl}(A_F) \simeq \mathbf{Z}$

**Proof.** If  $t = \sqrt{-a}$  is in  $k$ , then  $F = UV + YZ$  with  $U = X_1 + tX_2, V = X_1 - tX_2, Y = b(X_3 + tX_4), Z = X_3 - tX_4$  and therefore the assertion in this case is obvious by [F], § 14.

If  $t$  is not in  $k$ , then we can show that class  $\mathfrak{p}$ , where  $\mathfrak{p} = (x_1^2 + ax_2^2, x_3^2 + ax_4^2, x_1x_3 + ax_2x_4, x_1x_4 - x_2x_3)$  and  $x_i$  is the image of  $X_i$  in  $A_F$  ( $i=1, 2, 3, 4$ ), generates infinite cyclic group. Since we know that  $\text{Cl}(A_F)$  is a subgroup of an infinite cyclic group by the proof of Klein-Nagata theorem, we deduce that  $\text{Cl}(A_F) \simeq \mathbf{Z}$ . (Cf. the proof of L. Roberts quoted in [F], § 11 p. 52.)

**Lemma 2.** Let  $F = X_1^2 + aX_2^2 + bX_3^2 + cX_4^2$  ( $a, b, c \in k, abc \neq 0$ ). If none of  $-a, -bc, abc$  is the square of any element of  $k$ , then  $A_F$  is factorial.

**Proof.** In view of the proof of Klein-Nagata theorem, it is sufficient to show that  $G = bX_3^2 + cX_4^2$  is irreducible in  $k(t)[X_3, X_4]$ , where  $t = \sqrt{-a}$ . To do this it is sufficient to prove that  $-c/b$  cannot be written as the square of any element of  $k(t)$ . Assume the contrary, i.e., that it holds that

$$-c/b = (\alpha + \beta t)^2 = \alpha^2 - \beta^2 a + 2\alpha\beta t \quad (\alpha, \beta \in k)$$

Since 1 and  $t$  are linearly independent over  $k$  we must have  $2\alpha\beta = 0$ . Since we assumed that  $\text{ch } k \neq 2$  and since  $-c/b$  is not the square of any element of  $k$ , we have  $\beta \neq 0$  and therefore  $\alpha = 0$ . But then

$$-c/b = -a\beta^2 \quad \text{and} \quad abc = \beta^2 a^2 b^2,$$

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\*) In this note, the symbol [F] will refer to this literature, *Ergebn. Math. Bd. 74, Springer (1973)*.