58. *Asymptotic Distribution* mod m *and Independence of Sequences of Integers.* II

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This is the continuation of the paper on the preceding pages. For notation and terminology, we refer to the first part. The numbering of theorems, definitions, and equations is continued from the first part.

We remark that if (a_n) and (b_n) are independent mod m, then (a_n) and (a_n+b_n) need not be independent mod m. For, otherwise, since (a_n) and (0) are independent mod m by Theorem 4, (a_n) and (a_n) would be independent mod m, which happens only under special circumstances (see Theorem 3). However, the following result can be shown.

Theorem 7. *Let (a_n) and (b_n) be independent* mod m *with (b_n) u.d.* mod m. *Let $h, k, l \in \boldsymbol{Z}$ be such that* g.c.d. (l, m) *divides k. Then the sequences (ha_n), $n=1,2,\cdots$, and (ka_n+lb_n), $n=1,2,\cdots$, are independent* mod m.

Proof. Let $q \in \boldsymbol{Z}$ be a solution of the congruence $lx \equiv k \pmod m$. By a remark following Theorem 6, the sequence (qa_n+b_n), $n=1,2,\cdots$, is u.d. mod m. For $r, s \in \boldsymbol{Z}$ we have

$$\begin{aligned}\|A(a_n\equiv r, qa_n+b_n\equiv s)\| &= \|A(a_n\equiv r, b_n\equiv s-qr)\| \\ &= \|A(a_n\equiv r)\|\cdot\|A(b_n\equiv s-qr)\| = \|A(a_n\equiv r)\|\cdot\frac{1}{m} \\ &= \|A(a_n\equiv r)\|\cdot\|A(qa_n+b_n\equiv s)\|,\end{aligned}$$

and therefore the sequences (a_n) and (qa_n+b_n) are independent mod m. Thus, by Theorem 2, the sequences (ha_n) and (lqa_n+lb_n) are independent mod m. But the second sequence is mod m identical with (ka_n+lb_n), and so we are done.

Remark. Theorem 7 has the following partial converse. If (a_n) and (b_n) have α and β as their a.d.f. mod m, respectively, if $\alpha(j)>0$ and $\beta(j)>0$ for all j, and if (a_n) and (b_n) are independent mod m, then the independence mod m of (a_n) and (ka_n+lb_n) implies that g.c.d. (l,m) divides k. For if k were not divisible by g.c.d. (l,m), then we would have

$$\begin{aligned}\|A(a_n\equiv 0)\|\cdot\|A(ka_n+lb_n\equiv k)\| &= \|A(a_n\equiv 0, ka_n+lb_n\equiv k)\| \\ &= \|A(a_n\equiv 0, lb_n\equiv k)\| = 0.\end{aligned}$$

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