57. *Asymptotic Distribution* mod *m and Independence of Sequences of Integers.* I

By Lauwerens KUIPERS*) and Harald NIEDERREITER**)

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Let $m\geq 2$ be a fixed modulus. Let (a_n), $n=1,2,\cdots$, be a given sequence of integers. For integers $N\geq 1$ and j, let $A(N;j,a_n)$ be the number of n, $1\leq n\leq N$, with $a_n\equiv j \pmod m$. If

$$\alpha(j)=\lim_{N\to\infty} A(N;j,a_n)/N$$

exists for each j, then (a_n) is said to have α as its asymptotic distribution function mod m (abbreviated a.d.f. mod m). We denote $\alpha(j)$ also by $\|A(a_n\equiv j)\|$. Of course, it suffices to restrict j to a complete residue system mod m. If $\alpha(j)=1/m$ for $0\leq j<m$, then (a_n) is uniformly distributed mod m (abbreviated u.d. mod m) in the sense of Niven [4]. The numbers in brackets refer to the bibliography at the end of the second part of this paper.

If (b_n) is another sequence of integers, then for $N\geq 1$ and $j,k\in \boldsymbol{Z}$ we define $A(N;j,a_n;k,b_n)$ as the number of n, $1\leq n\leq N$, such that simultaneously $a_n\equiv j \pmod m$ and $b_n\equiv k \pmod m$. We write

$$(1)\qquad \|A(a_n\equiv j, b_n\equiv k)\|=\lim_{N\to\infty} A(N;j,a_n;k,b_n)/N$$

in case the limit exists. We note that if the limits (1) exist for all j, $k=0,1,\cdots,m-1$, then both (a_n) and (b_n) have an a.d.f. mod m. The following notion was introduced by Kuipers and Shiue [2].

Definition 1. The sequences (a_n) and (b_n) are called independent mod m if for all $j,k=0,1,\cdots,m-1$ the limits $\|A(a_n\equiv j, b_n\equiv k)\|$ exist and we have

$$\|A(a_n\equiv j, b_n\equiv k)\|=\|(A(a_n\equiv j)\|\cdot\|A(b_n\equiv k)\|.$$

Example 1. Let (c_n) be a sequence of integers that is u.d. mod m^2. Then writing $c_n\equiv a_n+mb_n \pmod{m^2}$, where $0\leq a_n<m$ and $0\leq b_n<m$, we obtain two sequences (a_n) and (b_n) that are independent mod m and u.d. mod m. See [2] and [1, Ch. 5, Example 1.5].

Example 2. Let α_1,α_2 be two real numbers such that $1,\alpha_1,\alpha_2$ are linearly independent over the rationals; or, more generally, let α_1,α_2 be two real numbers satisfying the condition of Theorem A in [3]. Then, according to this theorem, the sequence $(([n\alpha_1],[n\alpha_2]))$, $n=1,2,\cdots$, of lattice points is u.d. in $\boldsymbol{Z}^2$ (here $[x]$ denotes the integral part of

*) Department of Mathematics, Southern Illinois University, Carbondale, Illinois, U. S. A.

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