84. *Extremely Amenable Transformation Semigroups.* II

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Introduction. Let S be a semigroup and X a nonvoid set. Then we shall say that the pair (S, X) is a *transformation semigroup* if for every $s \in S$ there corresponds a map: $X \ni x \mapsto sx \in X$ such that $s(tx) = (st)x$ for all s, t in S and x in X. Let $B(X)$ be the Banach algebra of all real valued bounded functions on X with the supremum norm and $B(X)^*$ the conjugate Banach space of $B(X)$. For every $s \in S$ define the map $L_s: B(X) \to B(X)$ by $L_s f = {}_s f$ for $f \in B(X)$, where ${}_s f(x) = f(sx)$ for x in X. Then we have $L_s L_t = L_{ts}$ and $\|L_s\| \le 1$ for all s, t in S. The map $L: s \mapsto L_s$ is called the left regular antirepresentation of S on $B(X)$. $\varphi \in B(X)^*$ is a *mean* on $B(X)$ if $\inf\{f(x): x \in X\} \le \varphi(f) \le \sup\{f(x): x \in X\}$ for all $f \in B(X)$. If φ is a mean on $B(X)$, we have $\|\varphi\| = \varphi(I_X) = 1$ where I_X is the constant one function on X. $\varphi \in B(X)^*$ is called *invariant* if $\varphi({}_s f) = \varphi(f)$ for all $(s, f) \in S \times B(X)$. $\varphi \in B(X)^*$ is *multiplicative* if $\varphi(f \circ g) = \varphi(f) \cdot \varphi(g)$ for all $f, g \in B(X)$. By βX denote the set of all multiplicative means on $B(X)$, which is a w^*-compact subset of $B(X)^*$. For every $x \in X$ define $\delta_x \in \beta X$ by $\delta_x(f) = f(x)$ for all $f \in B(X)$ and denote by δ the map: $X \ni x \mapsto \delta_x \in \beta X$. Now we shall say a transformation semigroup (S, X) is *extremely amenable* if there is a multiplicative invariant mean on $B(X)$.

On extremely amenable transformation semigroups they are investigated by E. Granirer in [2] and by the author in [6]. In this paper, using the results in [2] and [6], we shall give various characterizations of extremely amenable transformation semigroups by means of the so-called *"fixed-point property"*, *"multiplicative invariant extension property"* and *"Reiter-Glicksberg's inequality"*. In § 4 we note addenda to my papers [6] and [7].

§ 1. Fixed-point property. We say a transformation semigroup (S, X) has a *fixed-point* if there is some x_0 in X such that $sx_0 = x_0$ for all $s \in S$. A transformation semigroup (S, Z) is called *compact* if Z is a compact Hausdorff space and for every $s \in S$ the map: $Z \ni z \mapsto sz \in Z$ is continuous. For example, for every $(s, \varphi) \in S \times \beta X$ define $s\varphi \in \beta X$ by $s\varphi(f) = \varphi({}_s f)$ for $f \in B(X)$. Then $(S, \beta X)$ is compact. Clearly (S, X) is extremely amenable if and only if $(S, \beta X)$ has a fixed-point. Let (S, X) and (S, Y) be transformation semigroups. A map $\sigma: X \to Y$ is called a *homomorphism* of (S, X) to (S, Y) if $s\sigma(x) = \sigma(sx)$ for all $(s, x) \in S \times X$.