82. *The Connection between the Order and the Diameter of a Neighborhood in a Vector Space*

By Masatoshi NAKAMURA
Kobe University

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In this paper, we investigate the connection between the order and the convergence exponent of the diameter of a bounded set in a normed space. We apply then the obtained results to a locally convex topological vector space.

1. Let E be a vector space over the field of real or complex numbers and A and B arbitrary sets in E.

For each positive number ε, let $M(A,B;\varepsilon)$ be the supremum of all natural numbers m, for which there exist elements $x_1,\cdots,x_m\in A$ with $x_i-x_j\notin\varepsilon B$ for $i\neq j$ $(1\leqq i, j\leqq m)$. Let $\rho(A,B)$ be the infimum of all positive numbers ρ, for which there is a positive number ε_0 such that $M(A,B;\varepsilon)<\exp(\varepsilon^{-\rho})$ for $0<\varepsilon<\varepsilon_0$. If no number ρ with the given property exists we set $\rho(A,B)=+\infty$. We then call $\rho(A,B)$ the *order of A with respect to B*; as is easily seen, we have

$$\rho(A,B)=\overline{\lim_{\varepsilon\to 0}}\{\log\log M(A,B;\varepsilon)/\log\varepsilon^{-1}\}.$$

The infimum $\delta_n(A,B)$ of all positive numbers δ, for which there is a vector subspace F of E of dimension at most n with $V\subset\delta U+F$ is called the *n-th diameter of A with respect to B.*

Let $a_1,a_2,\cdots$ be a sequence of positive numbers converging to zero. We call the infimum λ, of those values μ for which the series $\sum_{n=1}^{\infty}a_n^{\mu}$ converges, the *exponent of convergence* of the sequence $\{1/a_n\}$, and we call the exponent of convergence of the sequence $\{\log a_n^{-1}\}$ the *convergence type* of the sequence $\{a_n\}$. Let ε be a positive number, then we have the following two lemmas.

Lemma 1. *Let λ be the exponent of convergence of the sequence $\{1/a_n\}$. Then $\lambda=\overline{\lim}_{\varepsilon\to 0}\{\log m(\varepsilon)/\log\varepsilon^{-1}\}$, where $m(\varepsilon)$ denotes the number of terms of the sequence $\{a_n\}$ which are greater than ε.*

For a proof see [1], p. 89.

Lemma 2. *Let τ be the convergence type of the sequence $\{a_n\}$. Then*

$$\tau=\overline{\lim_{\varepsilon\to 0}}\{\log m(\varepsilon)/\log\log\varepsilon^{-1}\}.$$

Proof. Applying Lemma 1 to the sequence $\{\log a_n^{-1}\}$, we see that $\tau=\overline{\lim}_{\delta\to 0}\{\log l(\delta)/\log\delta^{-1}\}$ $(\delta>0)$, where $l(\delta)$ is the number of terms of $\{\log a_n^{-1}\}$ greater than δ. But obviously $l(\delta)=m(e^{-1/\delta})$. Therefore