80. The Completion by Cuts of an M-symmetric Lattice

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It is well known that the completion by cuts of a modular lattice is not necessarily modular ([1], p. 127, Example 9). But the following question was open ([2], p. 55, Problem 4): Is the completion by cuts of an M-symmetric lattice M-symmetric? In this paper we will give a negative answer to this question by constructing an atomistic M-symmetric lattice whose completion by cuts is not M-symmetric.

Let E be an infinite set and let A, B, C, D be mutually disjoint subsets of E which are all infinite. We take a sequence of subsets $\{C_n\}$ of C which satisfies the following two conditions:

(1) $C=C_0\supset C_1\supset C_2\supset\cdots$ and $\bigcap_{n=1}^{\infty} C_n=\phi$ (empty).

(2) For every $n=1,2,\cdots$, the set $C_{n-1}-C_n$ is infinite.

Moreover, we take a sequence of subsets $\{D_n\}$ of D satisfying the same conditions, and we put $A_n=A\cup C_n$ and $B_n=B\cup D_n$. We denote by $\boldsymbol{F}$ the family of all finite subsets of E, and we put

$$\boldsymbol{L}=\{E, A_n\cup F, B_n\cup F, F;\ 1\leqq n<\infty, F\in \boldsymbol{F}\}.$$

Proposition 1. *$\boldsymbol{L}$ forms an atomistic M-symmetric lattice, ordered by set-inclusion.*

Proof. It is evident that if $X, Y\in \boldsymbol{L}$ then their intersection $X\cap Y$ belongs to $\boldsymbol{L}$. Hence, the meet $X\wedge Y$ exists and equals to $X\cap Y$. If $X=A_m\cup F_1$ and $Y=B_n\cup F_2$ $(F_1, F_2\in \boldsymbol{F})$, then since E is the only upper bound of $\{X, Y\}$ in $\boldsymbol{L}$, the join $X\vee Y$ is E. Hence, $X\vee Y$ exists for every $X, Y\in \boldsymbol{L}$ and it holds that

(3) $$X\vee Y=\begin{cases} X\cup Y & \text{if } X\cup Y\in \boldsymbol{L}\\ E & \text{if } X\cup Y\notin \boldsymbol{L}.\end{cases}$$

Thus, $\boldsymbol{L}$ is a lattice and evidently it is atomistic. Next, we shall show that

(4) $(X, Y)M$ in $\boldsymbol{L}$ if and only if $X\cup Y\in \boldsymbol{L}$.

($(X, Y)M$ means that the pair (X, Y) is modular. See [2], (1.1).) If $X\neq E, Y\neq E$ and $X\cup Y\in \boldsymbol{L}$, then for any $X_1, Y_1\in \boldsymbol{L}$ with $X_1\leqq X$ and $Y_1\leqq Y$ we have $X_1\cup Y_1\in \boldsymbol{L}$. Hence, if $Y_1\leqq Y$ in $\boldsymbol{L}$, then

$$(Y_1\vee X)\wedge Y=(Y_1\cup X)\cap Y=Y_1\cup(X\cap Y)=Y_1\vee(X\wedge Y).$$

Hence, $(X, Y)M$. To prove the converse, it suffices to show that if $X=A_m\cup F_1$, $Y=B_n\cup F_2$ then the pairs (X, Y) and (Y, X) are not modular. Put $Y_1=B_{n+1}$. Then $Y_1\leqq Y$, and since $Y_1\vee X=E$ by (3) we