76. On Symbols of Fundamental Solutions of Parabolic Systems

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Introduction. The calculus of multiple symbols which has been developed in Kumano-go [1] enables us to construct the fundamental solution of parabolic equations only by symbol calculus (see C. Tsutsumi [4]). The purpose of the present paper is to show that a formal fundamental solution of a parabolic system has an asymptotic expansion in a class of pseudo-differential operators (§ 2) and to construct a fundamental solution with the same expansion (§ 3). The method of construction is the same as one used in C. Tsutsumi [4] for single equations.

1. Notations and a lemma. We shall denote by $S^m_{\rho,\delta}$ where $-\infty<m<+\infty$ and $0\leqq\delta<\rho\leqq 1$, the set of all $M\times M$ matrices $p(x,\xi)$ with components $p_{ij}(x,\xi)\in C^\infty(R^n_x\times R^n_\xi)$ which satisfy the inequality

$$|p^{(\alpha)}_{ij(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $p^{(\alpha)}_{ij(\beta)}(x,\xi)=\partial^\alpha_\xi D^\beta_x p_{ij}(x,\xi)$. We denote by $|p(x,\xi)|$ the norm of the matrix, that is,

$$|p(x,\xi)|=\sup_{0\neq y\in C^M}|p(x,\xi)y|/|y|$$

and define semi-norms $|p|_{m,k}$ by

$$|p|_{m,k}=\max_{|\alpha|+|\beta|\leqq k}\ \sup_{(x,\xi)}|p^{(\alpha)}_{(\beta)}(x,\xi)|\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}.$$

Then $S^m_{\rho,\delta}$ is a Fréchet space with these semi-norms. By $\mathcal{E}^0_t(S^m_{\rho,\delta})$ we denote a set of all matrices $p(t;x,\xi)\in S^m_{\rho,\delta}$ which are continuous with respect to parameter t for $0\leqq t\leqq T$. By w-$\mathcal{E}^0_{t,s}(S^m_{\rho,\delta})$ we denote a set of all matrices $p(t,s;x,\xi)\in S^m_{\rho,\delta}$ which are continuous with respect to parameter t and s for $0\leqq s\leqq t\leqq T$ with weak topology of $S^m_{\rho,\delta}$ defined as follows (see H. Kumano-go and C. Tsutsumi [2]): we say $\{p_j(x,\xi)\}^\infty_{j=0}\subset S^m_{\rho,\delta}$ converges weakly to $p(x,\xi)\in S^m_{\rho,\delta}$, if $\{p_j(x,\xi)\}^\infty_{j=0}$ is a bounded set of $S^m_{\rho,\delta}$ and $p^{(\alpha)}_{j(\beta)}(x,\xi)\to p^{(\alpha)}_{(\beta)}(x,\xi)$ as $j\to\infty$ uniformly on $R^m_x\times K$ for every α,β and compact set $K\subset R^n_\xi$.

When $p_\nu(x,\xi)\in S^{m_\nu}_{\rho,\delta}$, $\nu=1,2,\cdots,j$, we denote by $p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)$ the symbol of the product $P_1P_2\cdots P_j$ of pseudo-differential operators $P_\nu=p_\nu(x,D_x)$ which has the form (see Kumano-go [1])

$$\begin{aligned}&p_1(x,\xi)\circ p_2(x,\xi)\circ\cdots\circ p_j(x,\xi)\\ (1.1)\qquad &=Os\text{-}\int\cdots\int e^{-i(y^1\eta^1+\cdots+y^{j-1}\eta^{j-1})}p_1(x,\xi+\eta^1)p_2(x+y^1,\xi+\eta^2)\cdots\\ &\qquad\cdots p_j(x+y^1+\cdots+y^{j-1},\xi)dy^1\cdots dy^{j-1}d\eta^1\cdots d\eta^{j-1}\end{aligned}$$