# 101. On QF-Extensions in an H-Separable Extension

By Taichi NAKAMOTO
Okayama College of Science

(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1974)

Throughout the present note, $A/B$ will represent a ring extension with common identity 1, $V$ the centralizer $V_A(B)$ of $B$ in $A$, and $C$ the center of $A$. Following K. Hirata [2], $A/B$ is called an $H$-separable extension if $A\otimes_B A$ is $A$-$A$-isomorphic to an $A$-$A$-direct summand of a finite direct sum of copies of $A$. To be easily seen, $A/B$ is $H$-separable if and only if there exist some $v_i \in V$ $(i=1, \cdots, m)$ and casimir elements $\sum_j x_{ij}\otimes y_{ij}$ of $A\otimes_B A$ (which means $(\sum_j x_{ij}\otimes y_{ij})x = x(\sum_j x_{ij}\otimes y_{ij})$ for all $x\in A$) such that $\sum_{i,j} x_{ij}\otimes y_{ij}v_i = 1\otimes 1$ (cf. [4; Proposition 1]). Such a system $\{v_i;\sum_j x_{ij}\otimes y_{ij}\}_i$ will be called an $H$-system for $A/B$. On the other hand, $A/B$ is called a left $QF$-extension if ${}_BA$ is finitely generated (abbr. f.g.) projective and there exist some $f_r \in \mathrm{Hom}\,({}_BA_B, {}_BB_B)$ $(r=1, \cdots, n)$ and casimir elements $\sum_s c_{rs}\otimes d_{rs}$ of $A\otimes_B A$ such that $\sum_{r,s} c_{rs}f_r(d_{rs}) = 1$. Such a system $\{f_r;\sum_s c_{rs}\otimes d_{rs}\}_r$ will be called a left $QF$-system for $A/B$. Quite symmetrically, a right $QF$-extension and a right $QF$-system can be defined, and $A/B$ is called a $QF$-extension if $A/B$ is left $QF$ and right $QF$. One will easily see that $A/B$ is $QF$ if and only if there exist a left $QF$-system and a right $QF$-system for $A/B$.

The notion of an $H$-system will provide a new technique to reconstruct the commutor theory in $H$-separable extensions developed in [2], [3] and [5]. In this note, we use the technique to prove the following which are motivated by [4; Theorems 4 and 5]:

**Theorem 1.** *Assume that $A/B$ is an $H$-separable extension. Let $B'$ be an intermediate ring of $A/B$ with $V'=V_A(B')$ such that $V_A(V') = B'$ and ${}_{V'}V'_{V'} < \oplus_{V'}V_{V'}$ ($V'$ is a $V'$-$V'$-direct summand of $V$).*

(1) *If there exists a left (resp. right) $QF$-system for $A/B'$ then $V'/C$ is right (resp. left) $QF$.*

(2) *If there exists a right (resp. left) $QF$-system for $V'/C$ then $A_{B'}$ (resp. ${}_{B'}A$) is f.g. projective and there exists a left (resp. right) $QF$-system for $A/B'$.*

(3) *$A/B'$ is $QF$ if and only if so is $V'/C$.*

**Theorem 2.** *Assume that $A/B$ is an $H$-separable extension. Let $B'$ be an intermediate ring of $A/B$ with $V'=V_A(B')$ such that ${}_{B'}B'_{B'} < \oplus_{B'}A_{B'}$.*

(1) *If there exists a left (resp. right) $QF$-system for $B'/B$, then ${}_{V'}V$ (resp. $V_{V'}$) is f.g. projective and there exists a right (resp. left)*