98. *The Tensor Product of Weights*

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1. Introduction. The tensor product of normal semi-finite weights on von Neumann algebras was defined and used by several authors, e.g. F. Combes [3], A. Connes [6]. It was defined so that the resulting weight has favorable properties. Here in this note, we shall make a study on other possible definitions. We then establish a Radon-Nikodym type theorem for the tensor product of weights.

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2. The tensor product of normal semi-finite weights. Given a weight φ on a von Neumann algebra M, we denote by m_φ the *-subalgebra spanned by $n_\varphi^* n_\varphi$ where $n_\varphi=\{x \text{ in } M\,;\ \varphi(x^*x)<+\infty\}$. The linear extension on m_φ of $\varphi|_{(m_\varphi)_+}$ will be denoted by $\dot{\varphi}$. The following is the key lemma of our study.

Lemma 2.1. *Let a faithful normal semi-finite weight φ on M be given. Let τ be another normal semi-finite weight on M. If there exists a σ-weakly dense *-subalgebra B of m_φ, invariant under the modular automorphism group Σ of φ such that $\dot{\varphi}=\dot{\tau}$ on B, we have $\tau\leqq\varphi$, $\dot{\tau}|_{m_\varphi}=\dot{\varphi}$ and τ is faithful.*

Proof. The proof runs in the same way as in [5] Lemma 5.2. To get $\dot{\tau}|_{m_\varphi}=\dot{\varphi}$, we also make use of the expression $\tau(y^*x)=(\eta(x)\,|\,T\eta(y))$ for all x and y in n_φ as in [1] Lemma 2.3.

Let φ and ψ be faithful normal semi-finite weights on von Neumann algebras M and N. Let $\{\sigma_t\}$ and $\{\rho_t\}$ be the modular automorphism groups of φ and ψ, which will be denoted by Σ and Σ^ψ in what follows.

Proposition 2.2. *There exists a unique $\Sigma\otimes\Sigma^\psi$-invariant (i.e. $\sigma_t\otimes\rho_t$-invariant) normal semi-finite weight θ on $M\otimes N$ such that $\dot{\theta}\supset\dot{\varphi}\otimes_a\dot{\psi}$. Its modular automorphism group Σ^θ is the tensor product $\sigma_t\otimes\rho_t$. Furthermore if τ is a normal semi-finite weight on $M\otimes N$ such that $\tau\supset\dot{\varphi}\otimes_a\dot{\psi}$, we get $\dot{\tau}|_{m_\theta}=\dot{\theta}$ and τ is faithful.*

Proof. The existence is known ([6]. Definition 1.1.3). The uniqueness is due to [5] Proposition 5.9. Other properties are obtained from Lemma 2.1.

Theorem 2.3. *Let φ_1 and ψ_1 be normal semi-finite weights on M and N, p and q the support projections of φ_1 and ψ_1. There exists a unique normal semi-finite weight θ_1 on $M\otimes N$ such that $\dot{\theta}_1\supset\dot{\varphi}_1\otimes_a\dot{\psi}_1$ and θ_1 is $\Sigma^{\varphi_1}\otimes\Sigma^{\psi_1}$-invariant on the von Neumann algebra $p\otimes q(M\otimes N)p\otimes q$.*