97. A Remark on Quasi-Invariant Measure

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In this short note, we shall give a simple proof of Dao-Xing's theorem concerning quasi-invariant measures [1]. Although essential part of this proof is nearly equal to the original one, the assumptions of the original theorem are slightly weakened by this proof. Let $E \subset F$ be linear topological spaces and the linear topological space E be second category with countable basis of nbds (neighbourhoods) of 0, and let the injection map $E \to F$ be continuous.

Theorem. *Let $\mathfrak{B}$ be a σ-algebra of F which is invariant under E and contains all cylinder sets induced by F^* (dual of F). If μ is a non-trivial E-quasi-invariant measure on $(F, \mathfrak{B})$, then there exist a nbd V of 0 in E and a positive real number C such that*

$$\sup_{h \in V} |f(h)| \leqq C \int |f(x)|\, d\mu(x) \qquad \textit{for all } f \in F^*.$$

Proof. Assume the contrary. Then, for all positive integer n, we can find an $f_n \in F^*$ and a nbd V_n of 0 in E such that

$$\sup_{h \in V_n} |f_n(h)| > n \int |f_n(x)|\, d\mu(x),$$
$$V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots,$$

and $\{V_n\}$ is a basis of nbds of 0 in E.

Clearly we have $\int |f_n(x)|\, d\mu(x) < \infty$. We shall prove that $0 < \int |f_n(x)|\, d\mu(x)$. If not, $f_n(x) = 0$ almost everywhere on F. For $A_n = \{x \in F\,;\, f_n(x) = 0\}$, we have $\mu(A_n^c) = 0$.

Since μ is E-quasi-invariant, we have $\mu(A_n^c + h) = 0$ for all $h \in E$, i.e. $\mu([A_n \cap (A_n + h)]^c) = 0$.

Since μ is non-trivial, we have $\mu(A_n \cap (A_n + h)) > 0$. Hence, there exists $x \in F$ with $x \in A_n$ and $x - h \in A_n$. Thus, we have

$$f_n(h) = f_n(x) - f_n(x-h) = 0 \qquad \text{for all } h \in E.$$

This is a contradiction.

Let $a_n = \int |f_n(x)|\, d\mu(x)$ and consider $l^1(a_n) = \left\{\xi = (\xi_n)\,;\, \sum_n |\xi_n| \times \int |f_n(x)|\, d\mu(x) < \infty\right\}$. For $\xi = (\xi_n) \in l^1(a_n)$, we find

$$q_\xi(h) = \sum_n |\xi_n f_n(h)| < \infty \qquad \text{for all } h \in E$$

by the same argument which is already shown before.