On Characterizations of Spaces with $G_r$-diagonals

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A space $X$ is called to have a $G_r$-diagonal if the diagonal $\Delta$ in $X \times X$ is a $G_r$-set. A space $X$ is called to have a regular $G_r$-diagonal if $\Delta$ is a regular $G_r$-set, that is, $\Delta$ is written by the following:

$$\Delta = \cap \{U_n/n \in N\} = \cap \{U_n/n \in N\},$$

where $U_n$'s are open sets containing $\Delta$ in $X \times X$ and $N$ denotes the set of all natural numbers. Ceder in [1] characterized a $G_r$-diagonal as follows:

**Lemma 1.** A space $X$ has a $G_r$-diagonal if and only if there is a sequence $\{U_n/n \in N\}$ of open coverings of $X$ such that for each point $p$ in $X$

$$p = \cap \{S(p, U_n)/n \in N\}.$$

According to Zenor's result in [2], a regular $G_r$-diagonal is characterized as follows:

**Lemma 2.** A space $X$ has a regular $G_r$-diagonal iff there is a sequence $\{U_n/n \in N\}$ of open coverings of $X$ such that if $p, q$ are distinct points in $X$, then there are an integer $n$ and open sets $U$ and $V$ containing $p$ and $q$, respectively, such that no member of $\{U_n\}$ intersects both $U$ and $V$.

The object of the present paper is to characterize spaces with $G_r$- or regular $G_r$-diagonal by virtue of above lemmas as images of metric spaces under open mappings with some properties.

**Theorem 1.** A space $X$ has a $G_r$-diagonal iff there is an open mapping (single-valued) $f$ from a metric space $T$ onto $X$ such that

$$d(f^{-1}(p), f^{-1}(q)) > 0 \text{ for distinct points } p, q \in X.$$

**Proof.** Only if part: Define $T$ as follows:

$$T = \{(\alpha_1, \alpha_2, \cdots) \in N(A)/\cap \{U^\alpha_n/n \in N\} \neq \emptyset\},$$

where $\{U^\alpha_n = \{U^\alpha_n/\alpha \in A\}/n \in N\}$ is a sequence of open coverings of $X$ satisfying the condition in Lemma 1. If we define a mapping $f: T \to X$ as follows;

$$f(\alpha) = \cap \{U^\alpha_n/n \in N\} \text{ for } \alpha = (\alpha_1, \alpha_2, \cdots) \in T,$$

then $f$ is clearly a single-valued mapping from $T$ onto $X$. Since

$$f(N(\alpha_1, \cdots, \alpha_n)) = \cap \{U^\alpha_i/1 \leq i \leq n\},$$

it follows that $f$ is open. Let $p, q$ be distinct points in $X$; then by Lemma 1 we admit an integer $n$ in $N$ such that $q$ does not belong to $S(p, U_n)$. In this case it is proved that...