136. Projective Modules and 3-fold Torsion Theories

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(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

Let R be a ring with identity and R-mod the category of unital left R-modules. A 3-fold torsion theory for R-mod is a triple $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ of classes of left R-modules such that both $(\mathfrak{T}_1, \mathfrak{T}_2)$ and $(\mathfrak{T}_2, \mathfrak{T}_3)$ are torsion theories for R-mod in the sense of Dickson [2]. A class $\mathfrak{T}_2$ for which there exist classes $\mathfrak{T}_1$ and $\mathfrak{T}_3$ such that $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ is a 3-fold torsion theory for R-mod will be called a TTF-class following Jans [3]. In this case, $\mathfrak{T}_1$-torsion submodule $t_1(M)$ and $\mathfrak{T}_2$-torsion submodule $t_2(M)$ coincide with $t_1(R)\cdot M$ and $r_M(t_1(R))$ respectively for any left R-module M (cf. [4, Lemma 2.1]), where $r_M(*)$ denotes the right annihilator of $*$ in M.

An idempotent two-sided ideal I of R determines three classes of left R-modules

$$\mathfrak{C}_I = \{{}_RM \mid IM = M\},$$
$$\mathfrak{T}_I = \{{}_RM \mid IM = 0\}$$

and

$$\mathfrak{F}_I = \{{}_RM \mid r_M(I) = 0\},$$

and $(\mathfrak{C}_I, \mathfrak{T}_I, \mathfrak{F}_I)$ is then a 3-fold torsion theory for R-mod. In this case, the $\mathfrak{C}_I$-torsion submodule and $\mathfrak{T}_I$-torsion submodule of a left R-module M coincide with IM and $r_M(I)$ respectively.

Recently, in his paper [1], Azumaya has proved that, among other things, for a 3-fold torsion theory $(\mathfrak{C}_I, \mathfrak{F}_I, \mathfrak{C}_I)$ determined by the trace ideal I of a projective R-module P, a necessary and sufficient condition for $\mathfrak{C}_I$ to be a TTF-class is that ${}_{R/l_R(I)}P$ is a generator for $R/l_R(I)$-mod. In this note we shall give a similar condition for $\mathfrak{F}_I$ to be a TTF-class and look at the result due to Azumaya again from our point of view. Throughout this note, R-modules will mean left R-modules and $l(*)$ $(r(*))$ will denote the left (right) annihilator for $*$ in R.

We shall begin with a lemma which is in need of later discussions.

Lemma 1. *Let I be a left ideal and K a right ideal in R. Then the following conditions are equivalent:*

(1) $I + K = R$.

(2) *For any R-module M, $IM = 0$ implies that $KM = M$.*

If this is the case and if we assume moreover that $IK = 0$, then

(3) *both I and K are idempotent two-sided ideals of R and $I = l(K)$ and $K = r(I)$, and*