134. *On Submodules over an Asano Order of a Ring*[*)]

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(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

1. Let R be a ring with unity quantity, and let $\mathfrak{o}$ be a regular maximal order of R. The term *ideal* means a non-zero fractional two-sided $\mathfrak{o}$-ideal in R. We shall use small German letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ with or without suffices to denote ideals in R. The inverse of an ideal $\mathfrak{a}$ will be denoted by $\mathfrak{a}^{-1}$, and $\mathfrak{a}^*$ will denote $\mathfrak{a}^{-1-1}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *quasi-equal* if $\mathfrak{a}^{-1}=\mathfrak{b}^{-1}$; in symbol: $\mathfrak{a}\sim\mathfrak{b}$. The term *submodule* means a two-sided $\mathfrak{o}$-submodule which contains at least one regular element of R. A submodule M is said to be *closed* if whenever $\mathfrak{a}\subseteq M$ implies $\mathfrak{a}^*\subseteq M$. It is then clear that every submodule is closed when the arithmetic holds for $\mathfrak{o}$ (cf. [1, § 2]). For any two closed submodules M_1 and M_2 we define a product $M_1\circ M_2$ to be the set-theoretical union of all ideals $(\sum_{i=1}^{n}\mathfrak{a}_i\mathfrak{b}_i)^*$ where $\mathfrak{a}_i\subseteq M_1$ and $\mathfrak{b}_i\subseteq M_2$ $(i=1,\cdots,n)$. Now the set G of all ideals $\mathfrak{a}$ such that $\mathfrak{a}=\mathfrak{a}^*$ forms a *commutative* group under the multiplication "$\circ$" defined by $\mathfrak{a}\circ\mathfrak{b}=(\mathfrak{a}\mathfrak{b})^*=(\mathfrak{a}^*\mathfrak{b}^*)^*$; because G is a (conditionally) complete l-group under the above multiplication and the inclusion (cf. p. 91 in [5]). Hence $M_1\circ M_2=M_2\circ M_1$, and if the ascending chain condition in the sense of quasi-equality holds for integral ideals, the set $\mathfrak{M}$ of all closed submodules forms a commutative l-semigroup under the above multiplication and the set-inclusion (cf. Lemmas 5.1 and 5.2 in [2]).

Let $\mathfrak{P}$ be the set of all prime ideals which are not quasi-equal to $\mathfrak{o}$, let $|\mathfrak{P}|$ be the cardinal number of $\mathfrak{P}$, and let $\mathbf{Z}_{-\infty}$ be the set-theoretical union of the rational integers $\mathbf{Z}$ and $-\infty$. Then the complete direct sum $\bigoplus_{\mathfrak{P}}\mathbf{Z}_{-\infty}$ ($|\mathfrak{P}|$-copies) of $\mathbf{Z}_{-\infty}$ is an l-semigroup under the addition $[m_{\mathfrak{p}}]+[n_{\mathfrak{p}}]=[m_{\mathfrak{p}}+n_{\mathfrak{p}}]$ and the partial order $[m_{\mathfrak{p}}]\succ[n_{\mathfrak{p}}]\Leftrightarrow m_{\mathfrak{p}}\leq n_{\mathfrak{p}}$ for all $\mathfrak{p}\in\mathfrak{P}$, where $m_{\mathfrak{p}}, n_{\mathfrak{p}}\in\mathbf{Z}_{-\infty}$. Let $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-\infty}$ be the set of all vectors $[m_{\mathfrak{p}}]$ such that $m_{\mathfrak{p}}\leq 0$ for almost all $\mathfrak{p}\in\mathfrak{P}$. Then it forms an l-subsemigroup of $\bigoplus_{\mathfrak{P}}\mathbf{Z}_{-\infty}$.

The aim of the present note is to prove the following

Theorem. *If the ascending chain condition in the sense of quasi-equality* (cf. p. 109 in [1]) *holds for integral ideals, the l-semigroup $\mathfrak{M}$ of all non-zero closed submodules is isomorphic to $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_{-\infty}$ as an l-semigroup. If in particular the arithmetic holds for $\mathfrak{o}$, the l-semigroup $\mathfrak{M}$ of all submodules* (*containing regular elements*) *is isomorphic to $\bigoplus_{\mathfrak{P}}^*\mathbf{Z}_-$ as an l-semigroup, and every submodule $M\in\mathfrak{M}$ is written as follows*:

*) Dedicated to professor Kiiti Morita on his 60th birthday.