133. On the Fundamental Units of Real Quadratic Fields

By Masakazu KUTSUNA
College of General Education, Chukyo University

(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

1. Let $Q(\sqrt{D})$, ($D>0$ square-free rational integer), be a real quadratic field and put $D=n^2+r$ ($-n<r\leqq n$). Then, if $4n\equiv 0 \pmod r$ holds, the fundamental unit $\varepsilon_D>1$ of $Q(\sqrt{D})$ is well known ([1]) and such a real quadratic field $Q(\sqrt{D})$ is called *R-D* type. On the other hand, for any given real quadratic field $Q(\sqrt{D})$, its fundamental unit can be calculated by the continued fraction expansion of $\sqrt{D}$.

In this note, we shall first describe the fundamental units of all real quadratic fields in a similar fashion to *R-D* type, and give next its relation between contiriued fraction expansion. Finally, we shall give a generalization of a result of Morikawa [3] concerned with these facts.

2. The following theorem is a generalization of a result of Degert [1]:

Theorem 1. *For any given positive square-free integer D, let v_0 be the least positive integer such that $v_0^2D=n_0^2+r_0$ holds with integers n_0, r_0 satisfying $-n_0<r_0\leqq n_0$ and $4n_0\equiv 0 \pmod{r_0}$. Then the fundamental unit $\varepsilon_D>1$ of $Q(\sqrt{D})$ is of the following form:*

$\varepsilon_D=n_0+v_0\sqrt{D}$, $N\varepsilon_D=-\operatorname{sgn} r_0$ for $|r_0|=1$, (except for $D=5$, $v_0=1$),

$\varepsilon_D=(n_0+v_0\sqrt{D})/2$, $N\varepsilon_D=-\operatorname{sgn} r_0$ for $|r_0|=4$,

$\varepsilon_D=[(2n_0^2+r_0)+2n_0v_0\sqrt{D}]/|r_0|$, $N\varepsilon_D=1$ for $|r_0|\neq 1, 4$.

Remark. In the special case of $v_0=1$, this result coincides with Degert's.

Proof. Let $\varepsilon_D=(t_0+u_0\sqrt{D})/2$ be the fundamental unit of $Q(\sqrt{D})$ and ε_1 be the right-hand side of a formula for ε_D in Theorem 1. Then, it is easily shown that $u_0^2D=t_0^2\mp 4$, $4t_0\equiv 0 \pmod 4$ and that ε_1 is a unit of $Q(\sqrt{D})$. Here, if we suppose $\varepsilon_D\neq\varepsilon_1$, then it yields a contradiction. For, in the case of $|r_0|>4$, we get

$$\varepsilon_1=[(2n_0^2+r_0)+2n_0v_0\sqrt{D}]/|r_0|\geqq\varepsilon_D^2=(t_0^2\pm 2+t_0u_0\sqrt{D})/2.$$

Hence, we have $n_0v_0>t_0u_0$. On the other hand, since v_0 is the least positive integer such that $v_0^2D=n_0^2+r_0$, $-n_0<r_0\leqq n_0$, $4n_0\equiv 0 \pmod{r_0}$, we get $v_0<u_0$ and $n_0<t_0$, hence we have $n_0v_0<t_0u_0$. This is a contradiction. In other cases, we can easily induce contradiction similarly.

3. For any given D, it is generally difficult to find v_0 in Theorem 1, but if we use the continued fraction expansion of $\sqrt{D}$, v_0 is easily