132. *On Sylow Subgroups and an Extension of Groups*

By Zensiro GOSEKI
Gunma University

(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

Let A and B be groups. If there are homomorphisms f and g such that a sequence $\xrightarrow{f} A \xrightarrow{g} B \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{f}$ is exact, then we denote this collection by $(A, B: f, g)$ and we say $(A, B: f, g)$ to be *well defined*. Let $(A, B: f, g)$ and $(C, D: f_1, g_1)$ be well defined. If C and D are subgroups of A and B, respectively, and if $f=f_1$ on C and $g=g_1$ on D, then we call $(C, D: f_1, g_1)$ a *subgroup* of $(A, B: f, g)$ and in this case, we denote $(C, D: f_1, g_1)$ by $(C, D: f, g)$. Furthermore, we call $(C, D: f, g)$ a *normal subgroup* of $(A, B: f, g)$ if $C \lhd A$ and $D \lhd B$, and a *Sylow subgroup* of $(A, B: f, g)$ if C is a Sylow subgroup of A (in this case D is also a Sylow subgroup of B). We shall discuss the existence of such Sylow subgroups $(C, D: f, g)$ of $(A, B: f, g)$. It is easy to see that there are homomorphisms f and g such that $(A, B: f, g)$ is well defined iff there are groups M, N and homomorphisms $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that sequences $1 \to M \xrightarrow{\alpha_1} A \xrightarrow{\alpha_2} N \to 1$ and $1 \to N \xrightarrow{\beta_1} B \xrightarrow{\beta_2} M \to 1$ are exact. This shows that the results given in this note are related to an extension of groups.

Lemma 1. *Let $(A, B: f, g)$ be well defined. Let M and N be subgroups of A and B, respectively. Then $(M, N: f, g)$ is well defined iff $f(M)=f(A)\cap N$ and $g(N)=g(B)\cap M$.*

Proof. Since $(A, B: f, g)$ is well defined, $A/g(B) \cong f(A)$ and so $M/M\cap g(B) \cong Mg(B)/g(B) \cong f(M)$. Assume that $(M, N: f, g)$ is well defined. Then $M/g(N) \cong f(M)$. Hence $M/g(N) \cong M/M\cap g(B)$ where this isomorphism is given by $xg(N) \to x(M\cap g(B))$ for all $x \in M$. Hence $M\cap g(B)=g(N)$. Similarly $N\cap f(A)=f(M)$. Conversely, let $f(M) = N\cap f(A)$ and $g(N)=M\cap g(B)$. Then $M/g(N)=M/M\cap g(B) \cong Mg(B)/g(B) \cong f(M)$, i.e., $M/g(N) \cong f(M)$ where this isomorphism is given by $xg(N) \to f(x)$ for all $x \in M$. Similarly $N/f(M) \cong g(N)$ where this isomorphism is given by $yf(M) \to g(y)$ for all $y \in N$. Hence $(M, N: f, g)$ is well defined.

Lemma 2. *Let $(A, B: f, g)$ be well defined and let $(M, N: f, g)$ be a normal subgroup of $(A, B: f, g)$. Then $(A/M, B/N: \bar{f}, \bar{g})$ is well defined, where $\bar{f}$ and $\bar{g}$ are homomorphisms which are naturally induced by f and g, respectively.*

Proof. *By Lemma* 1, $f(A)\cap N=f(M)$. *Hence* $f^{-1}(N)=f^{-1}(f(A)$